Problem Set 2

1. $P(x) = A e^{-a(x-x_0)^2} dx$

Normalization: $\int_{-\infty}^{\infty} P(x)\,dx = 1$

$$\int_{-\infty}^{\infty} A e^{-a(x-x_0)^2} dx = 1$$

$u = x - x_0$
$du = dx$

$$A \int_{-\infty}^{\infty} e^{-au^2} du = 1$$

symmetric around $u=0$

$$2A \int_0^{\infty} e^{-au^2} du = 1$$

integral table

$$2A \left[\frac{1}{2}\sqrt{\frac{\pi}{a}}\right] = 1$$

$$A\sqrt{\frac{\pi}{a}} = 1$$

$$A = \sqrt{\frac{a}{\pi}}, \text{ so}$$

$$P(x) = \sqrt{\frac{a}{\pi}}\, e^{-a(x-x_0)^2}$$

2.

$$\langle x \rangle = \int_{-\infty}^{\infty} x\, P(x)\,dx = \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} x\, e^{-a(x-x_0)}\,dx$$

$u = x - x_0$
$du = dx$
$x = u + x_0$

$$= \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} (u + x_0)\, e^{-au^2} du$$

$$= \sqrt{\frac{a}{\pi}} \left[\underbrace{\int_{-\infty}^{\infty} u e^{-au^2} du}_{\text{odd function over symmetric limits } = 0} + x_0 \underbrace{\int_{-\infty}^{\infty} e^{-au^2} du}_{\text{even function over symmetric limits}} \right]$$

$$= \sqrt{\frac{a}{\pi}} \left[0 + 2x_0 \int_0^{\infty} e^{-au^2} du \right]$$

$$= \sqrt{\frac{a}{\pi}} \left[2x_0 \frac{1}{2} \sqrt{\frac{\pi}{a}} \right] = x_0$$

$$\boxed{\langle x \rangle = x_0}$$

more 2)

$$\langle x^2 \rangle = \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-a(x-x_0)^2} \, dx \qquad u = x - x_0 \qquad x = u + x_0$$
$$du = dx \qquad x^2 = (u+x_0)^2$$

$$= \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} (u+x_0)^2 e^{-au^2} \, du$$

$$= \sqrt{\frac{a}{\pi}} \left[\int_{-\infty}^{\infty} u^2 e^{-au^2} du + 2x_0 \underbrace{\int_{-\infty}^{\infty} u e^{-au^2} du}_{\text{odd function}} + x_0^2 \int_{-\infty}^{\infty} e^{-au^2} du \right]$$

$$= \sqrt{\frac{a}{\pi}} \left[2 \int_0^{\infty} u^2 e^{-au^2} du + 2x_0 \cdot 0 + 2x_0^2 \int_0^{\infty} e^{-au^2} du \right]$$

$$= \sqrt{\frac{a}{4\pi}} \left[2 \frac{1}{4a} \sqrt{\frac{\pi}{a}} + 0 + 2x_0^2 \frac{1}{2} \sqrt{\frac{\pi}{a}} \right]$$

$$= \frac{1}{2a} + 0 + x_0^2$$

$$\boxed{\langle x^2 \rangle = x_0^2 + \frac{1}{2a}}$$

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = x_0^2 + \frac{1}{2a} - x_0^2$$

$$\sigma_x^2 = \frac{1}{2a}$$

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\frac{1}{2a}}$$

$$\sigma_x = \sqrt{\frac{1}{2a}}$$

3. Normalization $\quad 1 = \int_0^\infty c e^{-\lambda x} dx$

$$= c\left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty$$

$$1 = c\left[0 + \frac{e^0}{\lambda}\right] = \frac{c}{\lambda}$$

$$\therefore \quad c = \lambda$$

$$\boxed{P(x) = \lambda e^{-\lambda x} dx}$$

$$\langle x \rangle = \int_0^\infty x \lambda e^{-\lambda x} dx = \lambda \int_0^\infty x e^{-\lambda x} dx$$

$$u = x \qquad dv = e^{-\lambda x} dx$$

$$du = dx \qquad v = -\frac{e^{-\lambda x}}{\lambda}$$

$$\langle x \rangle = \lambda\left[\frac{-x e^{-\lambda x}}{\lambda} + \frac{1}{\lambda}\int e^{-\lambda x} dx\right]$$

$$= \lambda\left[-\frac{x e^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2}\right]_0^\infty$$

$$= \lambda\left[-0 - 0 + 0 + \frac{1}{\lambda^2}\right] = \frac{1}{\lambda}$$

$$\boxed{\langle x \rangle = \frac{1}{\lambda}}$$

more 3) $\langle x^2 \rangle = \int_0^\infty x^2 \, \lambda e^{-\lambda x} \, dx$

$$= \lambda \int_0^\infty x^2 e^{-\lambda x} \, dx \quad \leftarrow \left[\begin{array}{ll} u = x^2 & dv = e^{-\lambda x} dx \\ du = 2x\,dx & v = \dfrac{-e^{-\lambda x}}{\lambda} \end{array}\right]$$

$$= \lambda \left[\frac{-x^2 e^{-\lambda x}}{\lambda} + \frac{2}{\lambda} \int x e^{-\lambda x} dx \right] \quad \leftarrow \left[\begin{array}{ll} u = x & dv = e^{-\lambda x} dx \\ du = dx & v = \dfrac{-e^{-\lambda x}}{\lambda} \end{array}\right]$$

$$= \lambda \left[\frac{-x^2 e^{-\lambda x}}{\lambda} + \frac{2}{\lambda} \left[\frac{-x e^{-\lambda x}}{\lambda} + \frac{1}{\lambda} \int e^{-\lambda x} dx \right] \right]$$

$$= \lambda \left[\frac{-x^2 e^{-\lambda x}}{\lambda} + \frac{2}{\lambda} \left[\frac{-x e^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right] \right]_0^\infty$$

$$= \left[-x^2 e^{-\lambda x} - \frac{2x}{\lambda} e^{-\lambda x} - \frac{2 e^{-\lambda x}}{\lambda^2} \right]_0^\infty$$

$$= \left[-0 - 0 - 0 + 0 + 0 + \frac{2}{\lambda^2} \right]$$

$$\langle x^2 \rangle = \frac{2}{\lambda^2}$$

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2}$$

$$\boxed{\sigma_x = \frac{1}{\lambda}}$$

5 a) $P(v_x) = \underbrace{c}_{\text{unknown proportionality const.}} e^{-mv_x^2/2k_BT}$

To normalize, the distribution must integrate to 1

$$1 = \int_{-\infty}^{\infty} P(v_x)\,dv_x = c\int_{-\infty}^{\infty} e^{-mv_x^2/2k_BT}\,dv_x$$

Gaussian integral: $\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \sqrt{\frac{\pi}{\alpha}}$ ← Learn this.

here, $\alpha = \frac{m}{2k_BT}$

$\therefore$

$$1 = c\sqrt{\frac{2\pi k_BT}{m}}$$

$$c = \sqrt{\frac{m}{2\pi k_BT}}$$

$$\boxed{P(v_x) = \sqrt{\frac{m}{2\pi k_BT}}\; e^{-mv_x^2/2k_BT}}$$

b) $\langle E\rangle = \left\langle \frac{mv_x^2}{2}\right\rangle = \frac{m}{2}\langle v_x^2\rangle$

$$= \frac{m}{2}\int_{-\infty}^{\infty} v_x^2 P(v_x)\,dv_x$$

$$= \frac{m}{2}\sqrt{\frac{m}{2\pi k_BT}}\int_{-\infty}^{\infty} v_x^2\, e^{-mv_x^2/2k_BT}\,dv_x$$ ← Look this integral up!

$$= \frac{m}{2}\sqrt{\frac{m}{2\pi k_BT}}\;\frac{\sqrt{\pi}}{2\left(\frac{m}{2k_BT}\right)^{3/2}} = \frac{m\sqrt{m}\sqrt{\pi}}{2\sqrt{2\pi k_BT}\,2}\;\frac{(2k_BT)^{3/2}}{m^{3/2}}$$

$$\boxed{\langle E\rangle = \frac{k_BT}{2}}$$ ← actually a statement of equipartition!

5 [redacted]c) $\langle v_x \rangle = \int_{-\infty}^{\infty} v_x \, P(v_x) \, dv_x$

$$= \sqrt{\frac{m}{2\pi k_B T}} \int_{-\infty}^{\infty} \underbrace{v_x}_{\substack{\text{odd} \\ \text{function}}} \underbrace{e^{-m v_x^2 / 2 k_B T} \, dv_x}_{\substack{\text{even} \\ \text{function}}}$$

symmetric limits

integrals of odd functions over symmetric limits are <u>always</u> 0

$$\boxed{\langle v_x \rangle = 0}$$

which is good, because any non-zero result would mean an ideal gas was in <u>motion</u> relative to the lab.

5 [redacted]d) $\langle m v_x \rangle = m \langle v_x \rangle = 0$

(there are other ways to do this, but this is the simplest!)

6. Channels are either open, with probability q or closed with probability $1-q$, so the expression we want is just the binomial distribution:

$$P(m,N) = q^m (1-q)^{N-m} \frac{N!}{m!\,(N-m)!}$$

7. We did most of this in class:

a) $$\frac{a_2}{a_1} = f(E_1 - E_2) \qquad \frac{a_3}{a_2} = f(E_2 - E_3)$$

$$\frac{a_3}{a_1} = f(E_1 - E_3)$$

$$\frac{a_3}{a_1} = \frac{a_2}{a_1}\frac{a_3}{a_2}$$

$$f(E_1 - E_3) = f(E_1 - E_2)\, f(E_2 - E_3)$$

Let $\left.\begin{aligned} x &= E_1 - E_2 \\ y &= E_2 - E_3 \end{aligned}\right\}$ $\begin{aligned} x+y &= E_1 - E_2 + E_2 - E_3 \\ &= E_1 - E_3 \end{aligned}$

$$f(x+y) = f(x)\,f(y) \qquad \text{QED!}$$

b) $f(x+y) = f(x)\,f(y)$

1

$$\ln[f(x+y)] = \ln[f(x)] + \ln[f(y)]$$

Take partial w.r.t. x on both sides:

$$\left[\frac{\partial \ln f(x+y)}{\partial x}\right]_y = \frac{\partial \ln f(x+y)}{\partial(x+y)}\left[\frac{\partial(x+y)}{\partial x}\right]_y = \frac{d \ln f(x+y)}{d(x+y)}$$

from LHS above

$$= \frac{d \ln f(x)}{dx}$$

from RHS above

Doing the same with y (keeping x fixed):

$$\left[\frac{\partial \ln f(x+y)}{\partial y}\right]_x = \frac{d \ln f(x+y)}{d(x+y)}\left[\frac{\partial(x+y)}{\partial y}\right]_x = \frac{d \ln f(x+y)}{d(x+y)}$$

$$= \frac{d \ln f(y)}{dy}$$

$$\therefore \quad \frac{d \ln f(x+y)}{d(x+y)} = \frac{d \ln f(x)}{dx} = \frac{d \ln f(y)}{dy}$$

1st set

2nd set

$$\therefore \quad \boxed{\frac{d \ln f(x)}{dx} = \frac{d \ln f(y)}{dy}} \qquad \text{Q.E.D.}$$

c) If both sides must be a constant

$$\frac{d \ln f(x)}{dx} = c \implies \frac{1}{f(x)}\frac{d f(x)}{dx} = c$$

$$\cancel{\frac{df(x)}{f(x)} = c\,dx} \qquad \frac{df(x)}{f(x)} = c\,dx$$

7 c)

$$\ln f(x) = xc + \text{const}$$ ← integrating both sides

$$f(x) = e^{cx} \underbrace{e^{\text{const}}}_{\text{constant}}$$

$$\therefore \boxed{f(x) \propto e^{xc} \qquad \text{Q.E.D.}}$$

8. Let's Do the relevant integrals first:

$$p(v_x) \propto e^{-mv_x^2/2k_BT} = A e^{-mv_x^2/2k_BT}$$

A ← normalization const

$$\int_{-\infty}^{\infty} p(v_x)\, dv_x = 1$$

$$1 = A \int_{\infty}^{\infty} e^{-mv_x^2/2k_BT}\, dv_x \quad \leftarrow \text{Gaussian integral (look it up)}$$

$$1 = A \sqrt{\frac{2\pi kT}{m}}$$

$$A = \sqrt{\frac{m}{2\pi k_BT}}$$

$$p(v_x) = \sqrt{\frac{m}{2\pi k_BT}}\, e^{-mv_x^2/2k_BT}$$

$$\langle E \rangle = \langle \tfrac{1}{2}mv_x^2 \rangle = \tfrac{1}{2}m\langle v_x^2 \rangle$$

$$= \frac{1}{2}m \int_{-\infty}^{\infty} e^{-mv_x^2/2k_BT}\, v_x^2\, dv_x \times \sqrt{\frac{m}{2\pi k_BT}}$$

$$= \frac{1}{2}m \sqrt{\frac{m}{2\pi k_BT}} \times \sqrt{2\pi}\left(\frac{k_BT}{m}\right)^{3/2}$$

$$\langle E \rangle = \frac{k_BT}{2} \quad \leftarrow \text{this is the right answe according to equipartition}$$

Now, on to the confusing part: First normalize

$$p(E) = Ae^{-E/k_BT}$$

$$1 = A\int_0^{\infty} e^{-E/k_BT}\, dE = k_BT$$

$$A = \frac{1}{k_BT}$$

$$p(E) = \frac{1}{k_BT}\, e^{-E/k_BT}$$

Now, to do the average energy:

$$\langle E \rangle = \int_0^\infty E \, p(E) \, dE$$

$$= \int_0^\infty \frac{E}{k_B T} e^{-E/k_B T} dE$$

$$= \frac{1}{k_B T} \left[-e^{-E/k_B T} (E + k_B T) k_B T \right]_0^\infty$$

$$\langle E \rangle = k_B T \quad \longleftarrow \text{ so what's wrong?}$$

Whats missing is the degeneracy

$$\sum_{\text{states}} e^{-\beta E_{\text{state}}} = \sum_{\text{levels}} g_{\text{level}} \, e^{-\beta E_{\text{level}}}$$

In this case, the states are the particle velocities, v_x

the levels are the energies, E

We really need to do this

degeneracy of energy levels

$$\int_{-\infty}^{\infty} p(v_x) \, dv_x \longrightarrow \int_0^\infty g(E) \, p(E) \, dE$$

The easy way to get this

$$p(v_x) \, dv_x = \sqrt{\frac{m}{2\pi k_B T}} \, e^{-m v_x^2 / 2 k_B T} \, dv_x$$

$$E = \frac{1}{2} m v_x^2 \longrightarrow \quad v_x = \sqrt{\frac{2E}{m}}$$

$$dE = m v_x \, dv_x \qquad v_x^2 = \frac{2E}{m}$$

$$dv_x = \frac{1}{m v_x} dE$$

$$dv_x = \frac{1}{\sqrt{2mE}} dE$$

If $$\int_{-\infty}^{\infty} p(v_x)\, dv_x = \int_{0}^{\infty} g(E)\, p(E)\, dE$$

we have

$$\int_{-\infty}^{\infty} \sqrt{\frac{m}{2\pi k_B T}}\; e^{-m v_x^2/2k_B T}\, dv_x = 1$$

Substituting in from previous page

$$\int_{0}^{\infty} A \sqrt{\frac{m}{2\pi k_B T}}\; e^{-E/k_B T}\, \frac{1}{\sqrt{2mE}}\, dE = 1$$

$$\int_{0}^{\infty} A \sqrt{\frac{1}{4\pi k_B T E}}\; e^{-E/k_B T}\, dE = 1$$

$$A = 2$$

$$\therefore \quad g(E)\, p(E)\, dE = \sqrt{\frac{1}{\pi k_B T E}}\; e^{-E/k_B T}\, dE$$

$$g(E) \propto \frac{1}{\sqrt{E}}$$

Basically there are more "states" available at low energies than at high energies

1. P_n = probability of <u>no one</u> in a group of n people having the same birthday.

$= 1 - P(\geq 2$ people with the same birthday$)$

that's what we want

$P_1 = 1 = \frac{365}{365}$

$P_2 = \frac{364}{365}$ ← that last day will make the 2 people have the same b-day

$P_3 = \frac{364}{365} * \frac{363}{365}$ ← <u>joint</u> probability of 2 & 1 <u>not</u> sharing B-day <u>and</u> 3&1, 3&2 not sharing a Birthday

$P_4 = P_3 * \frac{362}{365}$

$\vdots$

$P_n = \frac{1}{365^n} \frac{365!}{(365-n)!}$

$\therefore$

$P = 1 - \frac{1}{365^n} \frac{365!}{(365-n)!}$ ← general formula.

for 15 people:

$$\boxed{P = 0.252901 \approx 25.3\%}$$

Cross over to 50% happens when $n = 23$:

$P = 50.7297\%$ when $n = 23$

6. Channels are either open, with probability q or closed with probability $1-q$, so the expression we want is just the binomial distribution:

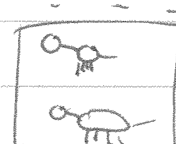
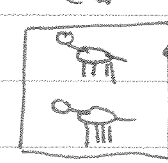
$$P(m, N) = q^m (1-q)^{N-m} \frac{N!}{m!\,(N-m)!}$$

4)

	Door 1	Door 2	Door 3
Scene 1	Car	$Goat_1$	$Goat_2$
Scene 2	Car	$Goat_2$	$Goat_1$
Scene 3	$Goat_1$	Car	$Goat_2$
Scene 4	$Goat_2$	Car	$Goat_1$
Scene 5	$Goat_1$	$Goat_2$	Car
Scene 6	$Goat_2$	$Goat_1$	Car

There are 6 scenarios (assuming distinguishable goats)
If we pick, say door **1**, Monty will have a reduced set of cases to show us a goat.

a) for scenes 1 & 2 he can show us either door 2 or 3
b) for scenes 3 & 4 he must show us door 3
c) for scenes 5 & 6 he must show us door 2

Scene	Door 1	Door 2		Door 3
1	Car	hidden goat	or	hidden goat
2	Car	hidden goat		hidden goat
3	hidden goat	Car		
4	hidden goat	Car		
5	hidden goat			Car
6	hidden goat			Car

current choice

revealed goats

4) continued

If we <u>stay</u> with door 1, our probability of landing the car happens in only 1/3 of scenarios (1 & 2)

If we <u>switch</u> to the ~~[illegible]~~ other door, our probability of landing the car happens in 2/3 of scenarios (3, 4, 5,

$\boxed{\text{We should switch doors, unless, of course we } \underline{\text{really}} \text{ wanted the goat.}}$

5) We did most of this in class:

a)

$$\frac{a_2}{a_1} = f(E_1 - E_2) \qquad \frac{a_3}{a_2} = f(E_2 - E_3) \qquad \frac{a_3}{a_1} = f(E_1 - E_3)$$

$$\frac{a_3}{a_1} = \frac{a_2}{a_1} \frac{a_3}{a_2}$$

$$f(E_1 - E_3) = f(E_1 - E_2)\, f(E_2 - E_3)$$

let $\left.\begin{aligned} x &= E_1 - E_2 \\ y &= E_2 - E_3 \end{aligned}\right\}$ $\begin{aligned} x + y &= E_1 - E_2 + E_2 - E_3 \\ &= E_1 - E_3 \end{aligned}$

$$f(x+y) = f(x)\, f(y) \qquad \text{Q.E.D.}$$

3. Channels are either open, with probability q or closed with probability $1-q$, so the expression we want is just the binomial distribution:

$$P(m, N) = q^m (1-q)^{N-m} \frac{N!}{m!(N-m)!}$$

(As an interesting challenge, can you derive an expression for the total average <u>current</u> through the membrane?)

4. We did most of this in class

a)

$$\frac{a_2}{a_1} = f(E_1 - E_2) \qquad \frac{a_3}{a_2} = f(E_2 - E_3) \qquad \frac{a_3}{a_1} = f(E_3 - E_1)$$

$$\frac{a_3}{a_1} = \frac{a_2}{a_1}\frac{a_3}{a_2}$$

$$f(E_1 - E_3) = f(E_1 - E_2)\, f(E_2 - E_3)$$

$$\left.\begin{aligned} \text{let } x &= E_1 - E_2 \\ y &= E_2 - E_3 \end{aligned}\right\} \quad \begin{aligned} x+y &= E_1 - E_2 + E_2 - E_3 \\ &= E_1 - E_3 \end{aligned}$$

$$f(x+y) = f(x)\,f(y) \qquad \text{Q.E.D.}$$