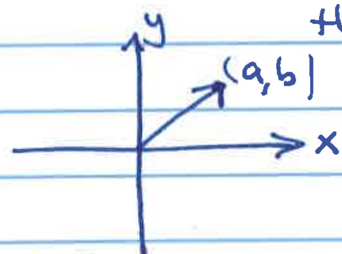
Linear Algebra

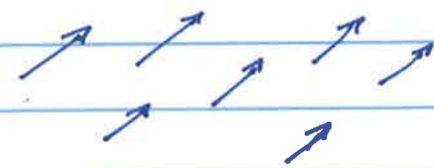
Scalars : quantities that have magnitude only
(time, temperature, mass)

vectors: an object with direction and magnitude
(velocity, force, dipole moment)

2D vector: $(a,b) = \vec{V}$ where a & b are called the components of the vector



All vectors that have the same length and direction are equal.



all equal (tail doesn't have to be at origin!)

Properties of vectors:

Length of vector $\vec{V} = V = |\vec{V}|$ = scalar

Equivalence of vectors:

$\vec{u} = (u_1, u_2)$ and $\vec{V} = (V_1, V_2)$ are equal if and only if $u_1 = V_1$ & $u_2 = V_2$.

Multiplication by a scalar: $c\vec{V} = (cV_1, cV_2)$

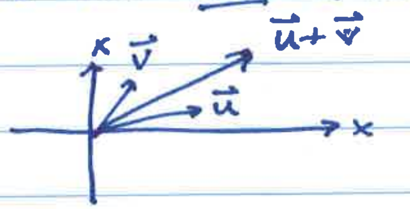
If $c \geq 0$, the length of $\vec{V}$ is changed, but the direction is not

If $c \leq 0$, the length is changed and the direction is reversed.

Example: $\vec{v} = (1, 1)$

$\vec{v}$ $\quad 2\vec{v}$ $\quad \frac{1}{2}\vec{v}$ $\quad -2\vec{v}$

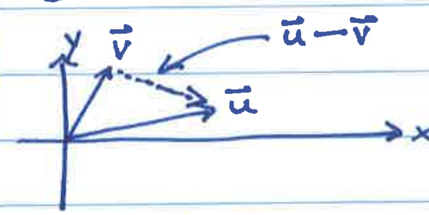
We can <u>add</u> 2 vectors



$\vec{u}+\vec{v} = (u_1+v_1, u_2+v_2)$

(note how this is like adding head-to-tail.

Or subtract them



$\vec{u}-\vec{v} = (u_1-v_1, u_2-v_2)$

(note how this is the ~~distance~~ vector between heads when tails are at same point)

<u>Unit vectors</u> (length of 1)

$\hat{\imath} = (1, 0)$ points along x axis

$\hat{\jmath} = (0, 1)$ points along y axis

Any 2D vector is an additive combination of $\hat{\imath}$ & $\hat{\jmath}$:

$$\vec{u} = (u_1, u_2) = (u_1, 0) + (0, u_2)$$

$$= u_1(1, 0) + u_2(0, 1)$$

$$= u_1\hat{\imath} + u_2\hat{\jmath}$$

$\vec{u}$ is often written as $u_x\hat{\imath} + u_y\hat{\jmath}$

$$u = |\vec{u}| = \sqrt{u_x^2 + u_y^2}$$

The Scalar product & magnitude

Multiplication of vectors can take different forms the simplest is the scalar or dot product:

vector • vector = scalar

The size of the scalar depends on the magnitude of the 2 vectors & their relative Direction

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$

the angle between the vectors

magnitudes of 2 vectors

What happens if we take the scalar product of a vector with itself?

$$\theta = 0, \qquad \cos\theta = 1$$

$$\vec{u} \cdot \vec{u} = |\vec{u}||\vec{u}| \cos\theta = |\vec{u}|^2$$

$$\therefore \quad |\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$$

← an alternative way to define the magnitude

One thing we can prove (but we won't) is that this means: $|\vec{u}| \geq 0$ and

$|\vec{u}| = 0$ implies $\vec{u} = (0, 0)$

(some subtlety will be required for complex vectors)

Scalar products behave like multiplication:

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

$$\vec{a} \cdot (n\vec{b}) = n\, \vec{a} \cdot \vec{b}$$

For the unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$, we have $|\hat{\imath}| = |\hat{\jmath}| = |\hat{k}| = 1$

and $|\hat{\imath}|^2 = |\hat{\jmath}|^2 = |\hat{k}|^2 = 1$

The unit vectors are perpendicular to each other so $\theta = \frac{\pi}{2}$. this means

$$\hat{\imath} \cdot \hat{\jmath} = |\hat{\imath}|\,|\hat{\jmath}| \cos\theta = 1 \times 1 \times 0 = 0$$

and likewise for $\hat{\imath} \cdot \hat{k}$ and $\hat{\jmath} \cdot \hat{k}$.

This means that if we describe vectors interms of their coordinates

$$\vec{u} = u_x \hat{\imath} + u_y \hat{\jmath} + u_z \hat{k}$$
$$\vec{v} = v_x \hat{\imath} + v_y \hat{\jmath} + v_z \hat{k}$$

We can expand out the full 3x3 possible terms:

$$\begin{aligned}\vec{u} \cdot \vec{v} = {} & u_x v_x \hat{\imath} \cdot \hat{\imath} + u_x v_y \hat{\imath} \cdot \hat{\jmath} + u_x v_z \hat{\imath} \cdot \hat{k} \\ & + u_y v_x \hat{\jmath} \cdot \hat{\imath} + u_y v_y \hat{\jmath} \cdot \hat{\jmath} + u_y v_z \hat{\jmath} \cdot \hat{k} \\ & + u_z v_x \hat{k} \cdot \hat{\imath} + u_z v_y \hat{k} \cdot \hat{\jmath} + u_z v_z \hat{k} \cdot \hat{k}\end{aligned}$$

Since only the diagonal terms are non-zero, we get:

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z$$

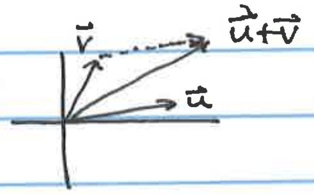
For an N-dimensional vector, you might write:

$$\vec{u} \cdot \vec{v} = \sum_{a=1}^{N} u_a v_a$$

Back to length: In 2-D:

$$\vec{c} = (a, b)$$
$$|\vec{c}| = \sqrt{\vec{c} \cdot \vec{c}}$$
$$= \sqrt{(a,b) \cdot (a,b)} = \sqrt{a^2 + b^2}$$

Back to vector addition for a moment



$$|\vec{u}+\vec{v}| = \sqrt{(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})}$$

$$= \sqrt{\vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}}$$

$$= \sqrt{|u|^2 + 2|\vec{u}||\vec{v}|\cos\theta + |\vec{v}|^2}$$

Since $\cos\theta \le 1$, we can make an inequality:

$$|\vec{u}+\vec{v}| = \sqrt{|\vec{u}|^2 + 2|\vec{u}||\vec{v}|\cos\theta + |\vec{v}|^2}$$

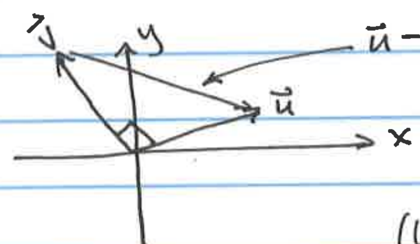
$$\le \sqrt{|\vec{u}|^2 + 2|\vec{u}||\vec{v}| + |\vec{v}|^2}$$

$$\le \sqrt{(|\vec{u}|+|\vec{v}|)^2}$$

$$|\vec{u}+\vec{v}| \le |\vec{u}| + |\vec{v}|$$ ← This is the triangle inequality.

The length of one side is always shorter than the combined lengths of the other sides.

Consider 2 vectors at right angles to each other



Pythagorean theorem

$$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2$$

$$(u_x - v_x)^2 + (u_y - v_y)^2 = (u_x^2 + u_y^2) + (v_x^2 + v_y^2)$$

$$u_x^2 - 2u_xv_x + v_x^2 + u_y^2 - 2u_yv_y + v_y^2 = u_x^2 + u_y^2 + v_x^2 + v_y^2$$

$$-2u_xv_x - 2u_yv_y = 0$$

$$u_xv_x + u_yv_y = 0$$

$$\vec{u}\cdot\vec{v} = 0$$ ← For 2 vectors at right angles $\vec{u}\cdot\vec{v} = 0$

perpendicular $\equiv$ orthogonal $\equiv \theta = \frac{\pi}{2} \equiv \vec{u}\cdot\vec{v} = 0$

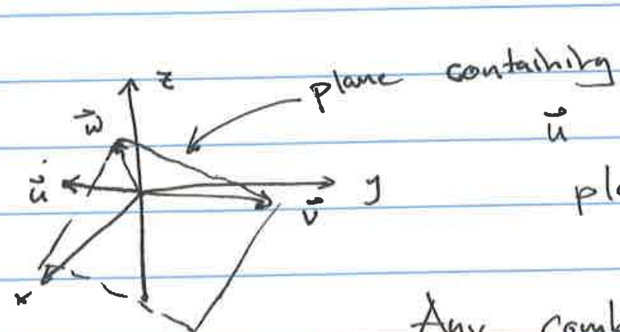
Dot products are useful in determining if 2 vectors are perpendicular:

$\vec{u} = (2,-1,1) \qquad \vec{v} = (3,5,-1) \qquad \vec{w} = (-1,-1,1)$

$$\begin{aligned}\vec{u}\cdot\vec{v} &= (2,-1,1)\cdot(3,5,-1)\\ &= 2\cdot 3 + (-1)\cdot 5 + 1\cdot(-1)\\ &= 6-5-1 = 0 \longrightarrow \vec{u}\ \&\ \vec{v} \text{ are } \perp\end{aligned}$$

$$\begin{aligned}\vec{v}\cdot\vec{w} &= (3,5,-1)\cdot(-1,-1,1)\\ &= 3(-1)+5(-1)+(-1)(1)\\ &= -3-5-1 = -9 \qquad \vec{v}\ \&\ \vec{w} \text{ are } \underline{\text{not}} \perp\end{aligned}$$

$$\begin{aligned}\vec{u}\cdot\vec{w} &= (2,-1,1)\cdot(-1,-1,1)\\ &= -2+1+1 = 0 \qquad \vec{u}\ \&\ \vec{w} \text{ are } \perp\end{aligned}$$



$\vec{u}$ is perpendicular to this plane!

Any combination of $\vec{w}$ & $\vec{v}$ are perpendicular to $\vec{u}$

$$\begin{aligned}\vec{u}\cdot(c_1\vec{v}+c_2\vec{w}) &= c_1\,\vec{u}\cdot\vec{v} + c_2\,\vec{u}\cdot\vec{w}\\ &= c_1\,(0) + c_2\,(0)\\ &= 0\end{aligned}$$

Can we find a vector that's perpendicular to both $\vec{u}$ & $\vec{v}$?

$$\underbrace{\vec{w}}_{\text{start with } \vec{w}}\,' = \vec{w} + \underbrace{c\vec{v}}_{\text{add a bit of } \vec{v}} \longleftarrow \text{still on the } \vec{w}, \vec{v} \text{ plane}$$

$$\vec{v} \cdot \vec{w}\,' = \vec{v} \cdot \vec{w} + c\,\vec{v} \cdot \vec{v} \qquad \longleftarrow \text{ we want this } = 0$$

$$0 = \vec{v} \cdot \vec{w} + c\,\vec{v} \cdot \vec{v}$$

$$c = \frac{-\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} = -\frac{-9}{(3,5,-1)\cdot(3,5,-1)}$$

$$= \frac{9}{9+25+1} = \frac{9}{35}$$

$$\vec{w}\,' = \vec{w} + c\vec{v} = (-1,-1,1) + \frac{9}{35}(3,5,-1)$$

$$\vec{w}\,' = \left(-\frac{8}{35}, \frac{10}{35}, \frac{26}{35}\right)$$

Let's check:

$$\vec{u} \cdot \vec{w}\,' = (2,-1,1) \cdot \left(-\frac{8}{35}, \frac{10}{35}, \frac{26}{35}\right) = \frac{-16}{35} - \frac{10}{35} + \frac{26}{35}$$
$$= 0 \quad \checkmark$$

$$\vec{v} \cdot \vec{w}\,' = (3,5,-1) \cdot \left(\frac{-8}{35}, \frac{10}{35}, \frac{26}{35}\right) = \frac{-24}{35} + \frac{50}{35} - \frac{26}{35}$$
$$= 0 \quad \checkmark$$

Why does this work?

We took $\vec{w}$ and removed some $\vec{v}$ from it, and the amount we removed was $\dfrac{\vec{w} \cdot \vec{v}}{|\vec{v}|^2}$

$\vec{w} \cdot \vec{v} / |\vec{v}|^2$ = projection of $\vec{w}$ onto $\vec{v}$.

The Vector or cross product

This takes 2 vectors and makes a third:

$$\vec{w} = \vec{u} \times \vec{v}$$

It has these properties:

- The magnitude of $\vec{w}$ is

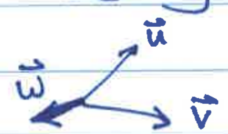
$$|\vec{w}| = |\vec{u}||\vec{v}| \sin\theta$$

where θ is the angle between $\vec{u}$ & $\vec{v}$.

This means $\vec{w}$ is longest when $\vec{u}$ & $\vec{v}$ are perpendicular ($\theta = \frac{\pi}{2} \rightarrow \sin\theta = 1$)

It also means $\vec{w}$ vanishes when $\vec{u}$ & $\vec{v}$ are parallel ($\theta = 0 \longrightarrow \sin\theta = 0$)

- $\vec{w}$ points in a direction perpendicular to the plane made by $\vec{u}$ & $\vec{v}$:



If $\vec{u}$ & $\vec{v}$ are in the plane of this page $\vec{w}$ is coming out of the page at you!

- Cross product distributes

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

and is linear:

$$\vec{a} \times (n\vec{v}) = \underset{\text{scalar}}{n}(\vec{a} \times \vec{v})$$

- Cross product does not commute!

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u} \quad \longleftarrow \text{direction reverses!}$$

and does not associate well:

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

Cross products of unit vectors:

$$\hat{\imath} \times \hat{\imath} = 0 \qquad (\text{also } \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0)$$

$$\left.\begin{aligned} \hat{\imath} \times \hat{\jmath} &= \hat{k} \\ \hat{\imath} \times \hat{k} &= -\hat{\jmath} \\ \hat{\jmath} \times \hat{k} &= \hat{\imath} \end{aligned}\right\}$$

> any triple is ⊕ if in ijk order — ijk, jki, kij
> and ⊖ if in another order — ikj, kji, jik
> cyclic permutations preserve ⊕
> skip this. It is confusing.

We can use these to decompose $\vec{u}$ & $\vec{v}$ and do a cross product:

$$\vec{u} = u_x \hat{\imath} + u_y \hat{\jmath} + u_z \hat{k} \qquad \vec{v} = v_x \hat{\imath} + v_y \hat{\jmath} + v_z \hat{k}$$

$$\begin{aligned} \vec{u} \times \vec{v} = {} & u_x v_x \, \hat{\imath} \times \hat{\imath} + u_x v_y \, \hat{\imath} \times \hat{\jmath} + u_x v_z \, \hat{\imath} \times \hat{k} \\ & + u_y v_x \, \hat{\jmath} \times \hat{\imath} + u_y v_y \, \hat{\jmath} \times \hat{\jmath} + u_y v_z \, \hat{\jmath} \times \hat{k} \\ & + u_z v_x \, \hat{k} \times \hat{\imath} + u_z v_y \, \hat{k} \times \hat{\jmath} + u_z v_z \, \hat{k} \times \hat{k} \end{aligned}$$

$$= (u_y v_z - u_z v_y)\hat{\imath} + (u_z v_x - u_x v_z)\hat{\jmath} + (u_x v_y - u_y v_x)\hat{k}$$

This is most conveniently done as a determinant:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \hat{\imath} \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} - \hat{\jmath} \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} + \hat{k} \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \hat{\imath}(u_y v_z - u_z v_y) - \hat{\jmath}(u_x v_z - u_z v_x) + \hat{k}(u_x v_y - u_y v_x)$$

Note $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{matrix} - \\ + \end{matrix} = ad - cb$

In determinants with more columns, we alternate sign as we go across the top.

Now, lets try one: $\vec{u} = (2,-1,1)$ $\vec{v} = (3,5,-1)$

$$\vec{u}\times\vec{v} = (2,-1,1)\times(3,5,-1)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 5 & -1 \end{vmatrix} = \begin{array}{l} \hat{\imath}\,((-1)(-1) - 5(1)) \\ -\hat{\jmath}\,((2)(-1) - 3(1)) \\ +\hat{k}\,(2\cdot 5 - 3(-1)) \end{array}$$

$$= \hat{\imath}(1-5) - \hat{\jmath}(-2-3) + \hat{k}(10+3)$$

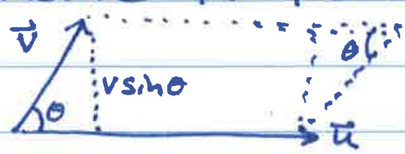
$$= -4\hat{\imath} + 5\hat{\jmath} + 13\hat{k}$$

$$\vec{u}\times\vec{v} = (-4,5,13)$$

(Note that the $\vec{w}'$ we found last time was:)

$$\left(\frac{-8}{35}, \frac{10}{35}, \frac{26}{35}\right) = \frac{2}{35}(-4,5,13)$$

A Geometric interpretation of $|\vec{w}|$:



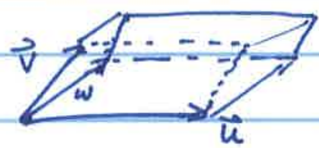
Area $= uv\sin\theta = |\vec{u}\times\vec{v}|$

The triple scalar product:

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = u_x(v_y w_z - v_z w_y) + u_y(v_z w_x - v_x w_z) + u_z(v_x w_y - v_y w_x)$$

$$= \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$\vec{u} \cdot (\vec{v} \times \vec{w})$ = volume of the parallelpiped formed by $\vec{u}, \vec{v}, \& \vec{w}$:



Vector Calculus

We are used to scalar functions of variables

$$f(x) = e^{-x}$$

we know $f(0) = 1$, $f(1) = 0.3$[illegible]7, etc.

Vector-valued functions have direction as well

$\vec{V}(t) \leftarrow$ the velocity of an atom as a function of time

$$\vec{v}(t) = (V_x(t), V_y(t), V_z(t))$$

Position is also a vector function of time:

$$\vec{r}(t) = (x(t), y(t), z(t))$$

We can take termwise derivatives of vector functions

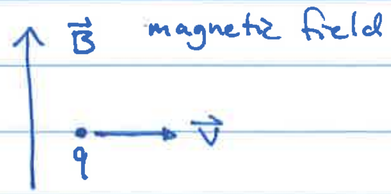
$$\vec{V}(t) = \frac{d}{dt} \vec{r}(t)$$

$$(V_x, V_y, V_z) = \frac{d}{dt}(x, y, z)$$

$$(V_x, V_y, V_z) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

We could also write a vector force in terms of a vector accelleration $F = ma$

$$\vec{F} = m \frac{d^2 \vec{r}}{dt^2}$$



Here's an example. The force on a charged particle moving through a magnetic field ($\vec{B}$) with velocity $\vec{v}$

$$\vec{F} = q\,\vec{v} \times \vec{B}$$

(in this illustration $\vec{F}$ comes out of the page.)

If the magnetic field points along the z-axis:

$$\vec{B} = \begin{pmatrix} 0 \\ 0 \\ B_z \end{pmatrix}$$ as it does in most NMR magnets

$$\vec{F} = m\vec{a}$$

$$q\,\vec{v} \times \vec{B} = m \frac{d}{dt}\vec{v}$$

$$q \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B_z \end{pmatrix} = m \begin{pmatrix} dv_x/dt \\ dv_y/dt \\ dv_z/dt \end{pmatrix}$$

$$q B_z \begin{pmatrix} v_y \\ -v_x \\ 0 \end{pmatrix} = m \begin{pmatrix} dv_x/dt \\ dv_y/dt \\ dv_z/dt \end{pmatrix}$$

this vector equation means the same as these 3 equations

$\therefore$

$$m \frac{dv_x}{dt} = q B_z v_y$$

$$m \frac{dv_y}{dt} = -q B_z v_x$$

$$m \frac{dv_z}{dt} = 0$$

this one is simple: $v_z(t) = v_z(0)$

Solving for $v_x(t)$ & $v_y(t)$ requires decoupling the 2 upper equations.

We start by taking the derivative of the 1^{st} one:

$$m \frac{dv_x}{dt} = q B_z v_y$$

$$m \frac{d^2 v_x}{dt^2} = q B_z \frac{dv_y}{dt}$$

Now, we plug the second equation $\left(\frac{dv_y}{dt} = -\frac{q B_z}{m} v_x \right)$ In to this:

$$m \frac{d^2 v_x}{dt^2} = q B_z \left(-\frac{q B_z}{m} v_x \right)$$

$$\frac{d^2 v_x}{dt^2} = -\left(\frac{q B_z}{m} \right)^2 v_x$$

$$\frac{d^2 v_x}{dt^2} + \left(\frac{q B_z}{m} \right)^2 v_x = 0$$

call $\frac{q B_z}{m} = \omega$

$$\frac{d^2 v_x}{dt^2} + \omega^2 v_x = 0$$

Which has solutions $v_x(t) = A e^{i\omega t} + B e^{-i\omega t}$

or $v_x(t) = C \cos \omega t + D \sin \omega t$

And $v_y(t)$ looks very similar!

Derivatives as vectors

The gradient is a vector operator

$$\vec{\nabla} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$$

So it takes a scalar function $\psi(x,y,z)$ and creates a vector of partial derivatives

$$\vec{\nabla} \psi(x,y,z) = \begin{pmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \\ \partial\psi/\partial z \end{pmatrix}$$

For any vector $\vec{v}$, the dot product

$$(\vec{\nabla} f) \cdot \vec{v}$$

is a <u>directional derivative</u> of the function f in the direction of $\vec{v}$.

In 2D, we have:

$$f_v = (\vec{\nabla} f) \cdot \vec{v} = v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y}$$

If the function is <u>not</u> changing along the $\vec{v}$ direction:

$$f_v = 0$$

We are then either:

- at a critical point so $\vec{\nabla} f = 0$

<u>or</u>

- the gradient is $\perp$ to $\vec{v}$, that is

$$(\vec{\nabla} f) \cdot \vec{v} = |\vec{\nabla} f| |\vec{v}| \cos\theta$$

with $\cos\theta = 0$

Whenever we map out contours of constant f, $\vec{\nabla} f$ is perpendicular to the contour lines.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$ ← Total differential

$$= (\vec{\nabla} f) \cdot \begin{pmatrix} dx \\ dy \end{pmatrix}$$ ← can also be related to the gradient

There are other important vector operators,

<u>div</u>, <u>grad</u>, <u>curl</u>

Divergence

$$\vec{\nabla} \cdot \vec{E}(x,y,z)$$

$\uparrow$ a vector function

$$\operatorname{div} \vec{E} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

$$= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \text{scalar}$$

Curl

$$\vec{\nabla} \times \vec{E}(x,y,z)$$

$$\operatorname{curl} \vec{E} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \text{vector}$$

The Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 \psi(x,y,z) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

$\uparrow$ scalar function

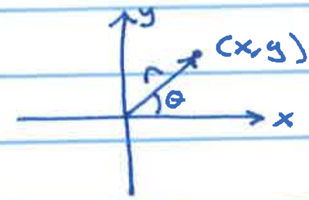
$$= \vec{\nabla} \cdot \left(\vec{\nabla} \psi\right) = \operatorname{div}(\operatorname{grad} \psi)$$

$$= \vec{\nabla} \cdot \vec{\nabla} \psi$$

Function	Result	
scalar	scalar	Laplacian – QM
Scalar	vector	Gradient – minimization
vector	scalar	Divergence · fluids, Gauss' Law
vector	vector	Curl · climate modeling

Non-cartesian coordinates

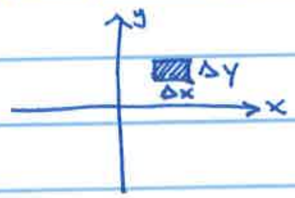
Plane-polar coordinates

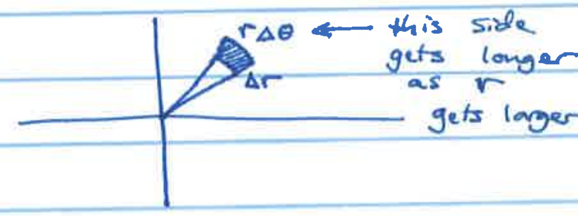


$x = r\cos\theta$ $\qquad r^2 = x^2 + y^2$ $\qquad 0 \le r \le \infty$

$y = r\sin\theta$ $\qquad \theta = \tan^{-1} y/x$ $\qquad 0 \le \theta < 2\pi$

To do integrals in polar coordinates we need the unit of area:





area unit		area unit
$\Delta x\, \Delta y$	$\longrightarrow$	$\Delta r \cdot r\,\Delta\theta = r\,\Delta r\,\Delta\theta$

Infinitessimal: $dx\,dy \longrightarrow r\,dr\,d\theta$

$$I = \iint dx\,dy \longrightarrow \iint r\,dr\,d\theta$$

Here's an example:

$$I = \int_{-\infty}^{\infty} e^{-ax^2}\,dx$$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-ax^2}\,dx\right)^2$$

$$= \left(\int_{-\infty}^{\infty} e^{-ax^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-ay^2}\,dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ax^2} e^{-ay^2}\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\; e^{-a(x^2+y^2)}$$

Now, to simplify: $r^2 = x^2 + y^2$

$dx\,dy \Rightarrow r\,dr\,d\theta$

$$I^2 = \int_0^\infty dr \int_0^{2\pi} d\theta \; r e^{-ar^2}$$

$$= 2\pi \int_0^\infty dr \; r e^{-ar^2}$$

$u = r^2$

$du = 2r\,dr \quad \Longrightarrow \quad r\,dr = \frac{du}{2}$

$$I^2 = 2\pi \int_0^\infty \frac{1}{2} e^{-u}\, du$$

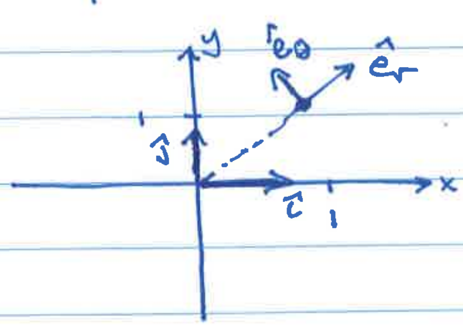
$$= \pi \left[\frac{-e^{-au}}{a} \right]_0^\infty = \pi \left(\frac{-e^{-\infty}}{a} + \frac{e^0}{a} \right)$$

$$= \frac{\pi}{a}$$

$\therefore$

$$\boxed{I = \int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}}$$

Vectors in plane coordinates

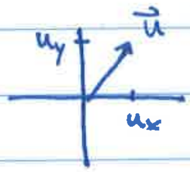


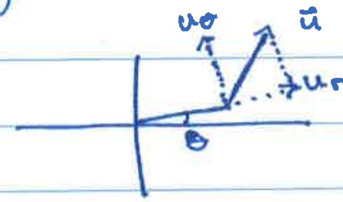
$\hat{e}_r$ & $\hat{e}_\theta$ are an orthogonal pair of unit vectors!

We can write any vector in either set of unit vectors or basis vectors:

$$\vec{u} = u_x \hat{i} + u_y \hat{j} = u_\theta \hat{e}_\theta + u_r \hat{r}$$

This can be a bit confusing at first:





(same vector

Cartesian

$\vec{u} = u_x \hat{\imath} + u_y \hat{\jmath}$

$u_x = \vec{u} \cdot \hat{\imath}$

$u_y = \hat{u} \cdot \hat{\jmath}$

$\vec{u} = u_r \hat{e}_r + u_\theta \hat{e}_\theta$

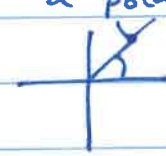
$u_r = \vec{u} \cdot \hat{e}_r$

$u_\theta = \vec{u} \cdot \hat{e}_\theta$

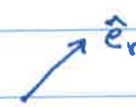
The transformation between cartesian & polar bases depends only on θ

$$\hat{e}_r = \cos\theta \, \hat{\imath} + \sin\theta \, \hat{\jmath}$$
$$\hat{e}_\theta = -\sin\theta \, \hat{\imath} + \cos\theta \, \hat{\jmath}$$

Example: Convert $\vec{u} = \hat{\imath} + 2\hat{\jmath}$ to a polar basis with $\theta = 45°$



$$\hat{e}_r = \cos(45)\,\hat{\imath} + \sin(45)\,\hat{\jmath}$$
$$= \frac{\sqrt{2}}{2}\,\hat{\imath} + \frac{\sqrt{2}}{2}\,\hat{\jmath}$$



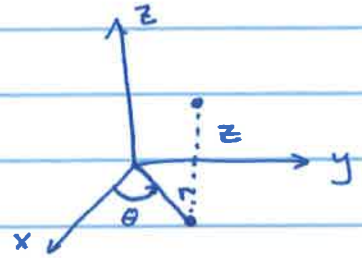
$$\hat{e}_\theta = -\sin(45)\,\hat{\imath} + \cos(45)\,\hat{\jmath}$$
$$= \frac{-\sqrt{2}}{2}\,\hat{\imath} + \frac{\sqrt{2}}{2}\,\hat{\jmath}$$

$$u_r = \vec{r} \cdot \hat{e}_r = (\hat{\imath} + 2\hat{\jmath}) \cdot \left(\frac{\sqrt{2}}{2}\hat{\imath} + \frac{\sqrt{2}}{2}\hat{\jmath}\right)$$
$$= \frac{\sqrt{2}}{2} + \frac{2\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$$

$$u_\theta = \vec{u} \cdot \hat{e}_\theta = (\hat{\imath} + 2\hat{\jmath}) \cdot \left(\frac{-\sqrt{2}}{2}\hat{\imath} + \frac{\sqrt{2}}{2}\hat{\jmath}\right) = \frac{\sqrt{2}}{2}$$

So: $\vec{u} = 1\hat{\imath} + 2\hat{\jmath} = \frac{3\sqrt{2}}{2}\hat{e}_r + \frac{\sqrt{2}}{2}\hat{e}_\theta$

Cylindrical coordinates



$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$

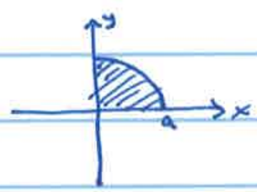
$y = r\sin\theta \qquad \theta = \tan^{-1} y/x$

$z = z \qquad z = z$

$$\iiint dx\,dy\,dz \longrightarrow \iiint r\,dr\,d\theta\,dz$$

Example: $I = \iiint_R x\,y\,z\; dx\,dy\,dz$

R is the region: $x \geq 0,\ y \geq 0,\ 0 \leq z \leq b$
and $x^2+y^2 \leq a^2$



$\leftarrow$ In cylindrical coordinates

$0 \leq r \leq a$

$0 \leq \theta \leq \pi/2$

$0 \leq z \leq b$

$$I = \iiint_R (r\,dr\,d\theta\,dz)(r\cos\theta)(r\sin\theta)(z)$$

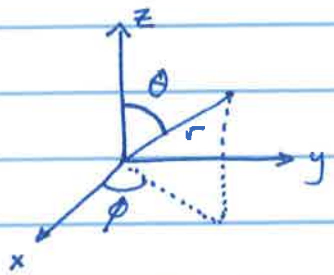
$$= \int_0^b z\,dz \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta \int_0^a r^3\,dr$$

$$= \left[\frac{z^2}{2}\right]_0^b \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^a$$

$$= \frac{b^2}{2} \quad \frac{1}{2} \quad \frac{a^4}{4}$$

$$I = a^4b^2/16$$

Spherical Coordinates



$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$r^2 = x^2 + y^2 + z^2$$
$$\cos\theta = \frac{z}{r}$$
$$\tan\phi = y/x$$

$$dx\,dy\,dz \;\to\; r^2\sin\theta\,dr\,d\theta\,d\phi$$

θ is an angle that measures "lattitude" or how far down from the north pole ($\oplus$ z axis) the vector $\vec{r}$ is. It goes from $\theta = 0$ (north pole) to $\theta = \pi$ (south pole)

ϕ measures longitude. It goes from $\phi = 0 \to \phi = 2\pi$

Example: $I = \iiint_R z^2\,dx\,dy\,dz$ where region R is

$$x^2 + y^2 + z^2 \le a^2$$ ← i.e. inside a sphere of radius a.

Region R in spherical coordinates:

$$0 \le r \le a$$
$$0 \le \theta \le \pi$$ ← important
$$0 \le \phi \le 2\pi$$

$$I = \iiint_R \left(r^2\sin\theta\,dr\,d\theta\,d\phi\right)\left(r\cos\theta\right)^2$$

$$= \int_0^a r^4\,dr \int_0^{2\pi} \cos^2\theta\sin\theta\,d\theta \int_0^{2\pi} d\phi$$

$$= \left[\frac{r^5}{5}\right]_0^a \left[-\frac{\cos^3\theta}{3}\right]_0^{2\pi} \left[\phi\right]_0^{2\pi}$$

$$I = \left(\frac{a^5}{5}\right)\left(-\frac{\cos^3\pi}{3} + \frac{\cos^3 0}{3}\right)(2\pi)$$

$$= \left(\frac{a^5}{5}\right)\left(\frac{2}{3}\right)(2\pi)$$

$$I = \frac{4\pi a^5}{15}$$

Determinants — we've seen these, but they are more useful

2×2: Consider 2 equations with 2 unknowns:

$$a_{11}x + a_{12}y = h_1$$
$$a_{21}x + a_{22}y = h_2$$

To solve this for x & y, we might try to eliminate y by cross multiplying by a_{12} and a_{22}:

$$a_{11}a_{22}x + a_{12}a_{22}y = h_1 a_{22}$$
$$a_{21}a_{12}x + a_{22}a_{12}y = h_2 a_{12} \quad \leftarrow \text{now subtract this guy}$$

$$(a_{11}a_{22} - a_{21}a_{12})x = h_1 a_{22} - h_2 a_{12}$$

$$x = \frac{h_1 a_{22} - h_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}}$$

Similarly:
$$y = \frac{h_1 a_{21} - h_2 a_{11}}{a_{11}a_{22} - a_{21}a_{12}}$$

Notice that the denominators are the same!

Also note that the denominator is a 2×2 determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

We can extend the idea of a determinant to more equations with more unknowns:

3x3 determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Systematic approach:

1. Choose a row or column (the row or column with the most 0s is the best choice) although we'll usually use the 1^{st} row:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \longleftarrow \text{this row}$$

2. For each element in the row, find its <u>minor</u> (the determinant formed by eliminating the row & column belonging to that element)

$$a_{11} \longrightarrow \underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\text{minor of } a_{11}} \longleftarrow M_{11}$$

$$a_{12} \longrightarrow \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \longleftarrow \text{minor of } a_{12}\ (M_{12})$$

$$a_{13} \longrightarrow \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \longleftarrow M_{13}$$

3. Find the cofactor of each element in your chosen row:

Cofactor of element $a_{ij} = (-1)^{i+j} M_{ij}$

where M_{ij} is the minor of element a_{ij}

$$(-1)^{i+j} \longrightarrow \begin{vmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & & & & & \end{vmatrix}$$

Here are the cofactors of the original matrix:

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

4. Multiply each element by its respective cofactor and add them up:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

5. Repeat steps 1–4 on the new determinants until only 2x2 remain. In this case, we're done.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{23} - a_{21}a_{33})$$

$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example:

$$\begin{vmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & -2 & 1 \end{vmatrix}$$

Let's pick 1st column for cofactor expansion this time.

$$= 2\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} - 0\begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= 2(3-2) - 0(-1+2) + 2(1-3)$$

$$= 2 \quad - 0 \quad - 4$$

$$= -2$$

Properties of Determinants:

1. The value of a determinant if the rows are made into columns in the same order: eg row1 → col 1, row2 → col2: This is called taking a transpose of a matrix:

$$\det(A) = \det(A^T)$$

$$\begin{vmatrix} 1 & 2 & 5 \\ -1 & 0 & -1 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 0 & 1 \\ 5 & -1 & 2 \end{vmatrix} = -6$$

2. If any 2 rows or columns are identical, the determinant is zero:

$$\begin{vmatrix} 4 & 2 & 4 \\ -1 & 0 & -1 \\ 3 & 1 & 3 \end{vmatrix} = 0$$

← columns 1 & 3 are the same.

3. If any 2 rows or columns are interchanged the sign of det (A) is reversed:

$$\begin{vmatrix} 3 & 1 & -1 \\ -6 & 4 & 5 \\ 1 & 2 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & -1 \\ 4 & -6 & 5 \\ 2 & 1 & 2 \end{vmatrix}$$

← cols 1 & 2 are swapped

4. Scalar multiplication works, but on a column-by-column or row-by-row basis

$$\begin{vmatrix} 6 & 8 \\ -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix}$$

$$12 + 8 = 2(6+4)$$

5) Sums & Differences in a single row or column can be written as 2 determinants

$$\begin{vmatrix} a_{11}+b & a_{12} & a_{13} \\ a_{21}+c & a_{22} & a_{23} \\ a_{31}+d & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b & a_{12} & a_{13} \\ c & a_{22} & a_{23} \\ d & a_{32} & a_{33} \end{vmatrix}$$

6) The value is unchanged if we add or subtract one row or column to another:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+b & b & c \\ d+e & e & f \\ g+h & h & i \end{vmatrix}$$

Likewise scalar mult + addition does nothing

$$\begin{vmatrix} a+kb & b & c \\ d+ke & e & f \\ g+kh & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + k \begin{vmatrix} b & b & c \\ e & e & f \\ h & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + k(0)$$

Rule 2.

Matrices

Vector transformations:

An operator is something that takes a vector and turns it into another vector

$$\hat{O}\,\vec{v} = \vec{w}$$

For example, reflection across the yz plane is an operator (call it $\hat{\sigma}$)

$$\hat{\sigma}\,(v_x, v_y, v_z) = (-v_x, v_y, v_z)$$

Or inversion through the origin:

$$\hat{I}\,(v_x, v_y, v_z) = (-v_x, -v_y, -v_z)$$

Or rotation by 45° around the z-axis:

$$\hat{R}_{\pi/4}\cdot(v_x, v_y, v_z) = \left[\tfrac{1}{\sqrt{2}}(v_x - v_y), \tfrac{1}{\sqrt{2}}(v_x + v_y), v_z\right]$$

Operators can create vectors of a different dimension for example Projection onto x-y plane:

$$\hat{P}_{xy}\,(v_x, v_y, v_z) = (v_x, v_y)$$

Most physically important operators are linear

$$\hat{O}(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1\hat{O}\vec{v}_1 + c_2\hat{O}v_2$$

The key property of an operator is what it does to the coordinates of a vector, e.g.

$$\hat{R}_{\pi/4}\,\hat{\imath} = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)$$

$$\hat{R}_{\pi/4}\,\hat{\jmath} = \left(\tfrac{-1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)$$

$$\hat{R}_{\pi/4}\,\hat{k} = (0, 0, 1)$$

← If we know what a linear operator does to the complete set of unit vectors, we can reconstruct $\hat{R}_{\pi/4}$

So: $$R_{\pi/4}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = R_{\pi/4}\, v_x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + v_y R_{\pi/4}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + v_z R_{\pi/4}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= v_x\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right) + v_y\left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right) + v_z(0, 0, 1)$$

$$= \left(\tfrac{1}{\sqrt{2}}(v_x - v_y), \tfrac{1}{\sqrt{2}}(v_x + v_y), v_z\right)$$

There are 9 numbers that tell us how this operator manipulates <u>any</u> vector. (in 3 dimensions)

We can extend this to any rotation operator around z-axis:

$$\hat{R}_\theta \hat{\imath} = (\cos\theta, \sin\theta, 0)$$
$$\hat{R}_\theta \hat{\jmath} = (-\sin\theta, \cos\theta, 0)$$
$$\hat{R}_\theta \hat{k} = (0, 0, 1)$$

$$\hat{R}_\theta\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = (v_x\cos\theta - v_y\sin\theta,\ v_x\sin\theta + v_y\cos\theta,\ v_z)$$

We typically write down these 9 numbers in a matrix:

$$\hat{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix corresponds <u>exactly</u> to the operator and we can write:

$$\hat{R}_\theta \cdot \vec{v} = w$$

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v_x\cos\theta - v_y\sin\theta \\ v_x\sin\theta + v_y\cos\theta \\ v_z \end{pmatrix}$$

A matrix vector multiplication looks like a sequence of 3 dot products:

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = v_x \cos\theta - v_y \sin\theta$$

And:

$$\begin{pmatrix} \sin\theta & \cos\theta & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = v_x \sin\theta + v_y \cos\theta$$

And:

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = v_z$$

Go horizontally across one row of the matrix and vertically down the vector to create each element of the new vector.

Matrices are not necessarily square. Here's a matrix for $\hat{P}_{xy}$, the projection of a vector onto the x-y plane

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$2 \times 3 \quad \cdot \quad 3 \times 1 \quad = \quad 2 \times 1$$

To do matrix-vector multiplication, we need the horizontal dimension of the matrix to be the same as the vertical dimension of the vector.

Vector dot products can be thought of as matrix vector multiplications:

$$\vec{u} \cdot \vec{v} = \begin{pmatrix} u_x & u_y & u_z \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = u_x v_x + u_y v_y + u_z v_z$$

$$1 \times 3 \quad \cdot \quad 3 \times 1 \quad = \quad 1 \times 1$$

must match

dimensions of result

Matrix multiplication

Reminder: Linear operators can be added together & multiplied by scalars

$$(\hat{O} + \hat{P})\vec{v} = \hat{O}\vec{v} + \hat{P}\vec{v}$$

$$(a\hat{O})\vec{v} = a(\hat{O}\vec{v})$$

These operations are done element-by-element on matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 3 \\ 1 & 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & -1-\lambda & 3 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

Multiple operations: Each operator has an action it does:

$\hat{R}_\theta \vec{u} = \vec{v}$: Rotate $\vec{u}$ by angle θ around z axis to get $\vec{v}$

$\hat{P}_{xy} \vec{v} = \vec{w}$: Project $\vec{v}$ onto x-y plane to get $\vec{w}$

We can put these together: $(\hat{P}_{xy} \hat{R}_\theta)\vec{u} = \vec{w}$

Rotate $\vec{u}$ by θ around z, then project result onto x-y plane.

e.g.

$$\hat{P}_{xy} \hat{R}_\theta \vec{u} = \hat{P}_{xy} (\hat{R}_\theta \vec{u})$$
$$= \hat{P}_{xy} (\vec{v})$$
$$= \vec{\omega}$$

$\hat{P}_{xy} \hat{R}_\theta$ is itself an operator as it takes one vector into another.

Operator multiplication (chaining) is not <u>necessarily</u> commutative. <u>Order</u> matters.

Multiplying matrices works just like matrix vector multiplication, but:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \end{pmatrix}$$

$$2\times 3 \quad \cdot \quad 3\times 3 \quad = \quad 2\times 3$$

match

dimension of result

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = \underline{\underline{C}}$$

$A_{ij} = [\underline{\underline{A}}]_{ij}$ ← element of matrix $\underline{\underline{A}}$ in i^{th} row & j^{th} column

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

In general

$$C_{ij} = \sum_k A_{ik} B_{kj} .$$

$$\underline{\underline{C}} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} & A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} & \cdots & \\ \vdots & & \end{pmatrix}$$

The Identity Matrix

In 3D:

$$\underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In any number of dimensions, $\underline{\underline{I}}$ is a matrix with 1's along the diagonals and 0's everywhere else

$$\underline{\underline{I}} \cdot \vec{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x + 0 + 0 \\ 0 + V_y + 0 \\ 0 + 0 + V_z \end{pmatrix} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$\underline{\underline{I}}$ takes any vector into itself. Likewise for matrices:

$$\underline{\underline{A}} \cdot \underline{\underline{I}} = \underline{\underline{I}} \cdot \underline{\underline{A}} = \underline{\underline{A}}$$

If 2 matrices multiply together to give the identity:

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = \underline{\underline{I}}$$

We say that $\underline{\underline{B}}$ is the inverse of $\underline{\underline{A}}$.

$$\underline{\underline{A}} \cdot \underline{\underline{A}}^{-1} = \underline{\underline{I}}$$

2x2 general formula for an inverse

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

what is in the denominator?
det( )

For bigger matrizes, the task is more difficult

Gaussian Elimination

(Basically the way you would solve a set of equations for unknowns)

Augmented Matrix: the matrix we have I

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

we'll turn this side into I — this side will contain A^{-1}

1) Subtract row 1 from row 2:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \end{array}\right)$$

2) Swap rows 2 & 3:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

3) Subtract 3× 1st row from 2nd row:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -4 & -2 & -3 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

4) Divide 2nd row by −4:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 3/4 & 0 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

← upper triangular, so almost done:

5) Subtract $\frac{1}{2}$ 3rd row from 2nd

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{5}{4} & -\frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

6) Subtract 3rd row from 1st:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 5/4 & -1/2 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

7) Subtract 2nd row from 1st:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & 5/4 & -1/2 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\underline{A}^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -2 & 1 \\ 5 & -2 & -1 \\ -4 & 4 & 0 \end{pmatrix}$$

Rotation Matrices invert easily

$$\underset{=}{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\underset{=}{R}_\theta^{-1} = \underset{=}{R}_{-\theta}$$

$$= \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Let's verify

$$\underset{=}{R}_\theta^{-1}\,\underset{=}{R}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underset{=}{I} \quad \checkmark$$

Note that $\underset{=}{R}_\theta^{-1} = (\underset{=}{R}_\theta)^T \leftarrow$ transpose (e.g. exchange rows & columns)

Reflections also invert easily

$$\underset{=}{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \underset{=}{\sigma}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \underset{=}{\sigma}$$

$$\underset{=}{\sigma}^{-1}\cdot\underset{=}{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{pmatrix} = \underset{=}{I} \quad \checkmark$$

Orthogonal Matrices

We've mentioned the transpose briefly (simply exchange rows & columns)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

Not as arbitrary as it seems $\underline{A} \cdot \underline{B}$ is the rows of $\underline{A}$ multiplying columns of $\underline{B}$ which leads us to:

$$(\underline{A} \cdot \underline{B})^T = \underline{B}^T \cdot \underline{A}^T$$

columns of $\underline{B}$ rows of $\underline{A}$

An orthogonal matrix has this property

$$\underline{A}^T = \underline{A}^{-1}$$

or

$$\underline{A}\,\underline{A}^T = \underline{A}^T \underline{A} = \underline{I}$$

If we write this out for a 3x3

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \cdot \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which means $\left((a_1, a_2, a_3) \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 1 \right.$

and also for b & c

while

$$(a_1 \quad a_2 \quad a_3) \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0$$

or:

$$\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c} = 0$$
$$\vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b} = \vec{c} \cdot \vec{c} = 1$$

The Rows (& Columns) of an orthogonal matrix are themselves orthogonal Vectors!

If you take your solutions to problem 7 and put them in a matrix, you will have an orthogonal matrix!