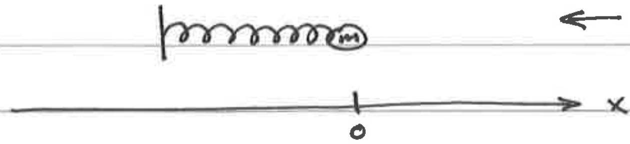
Ordinary Differential Equations

Let's start with an example:



← An idealized spring with an attached mass m.

The equilibrium position for the spring is defined arbitrarily at $x=0$.

Displacing the mass from $x=0$, will result in a restoring force. A perfect Hooke's law spring has a restoring force that is linear in the displacement:

$$F = -kx$$

← k is a spring constant

$-$ sign insures that restoring force opposes the displacement

How will the spring move? To answer this we need a simple statement of classical mechanics:

Newton's 2^{nd} law: $F = ma$

Here $v = \dfrac{dx}{dt}$ ← velocity

$a = \dfrac{dv}{dt}$ ← accelleration

$= \dfrac{d^2x}{dt^2}$

$\therefore$

$$F = m\frac{d^2x}{dt^2}$$

$$-kx = m\frac{d^2x}{dt^2}$$

Or:

$$m\frac{d^2x}{dt^2} + kx = 0$$

← This is a linear ordinary differential equation.

Ordinary: Has only one dependent variable (x) and one independent variable (t)

Linear: Contains only combinations of x & derivatives of x and not functions of x (like x^2 or $\sin(x)$). Also no functions of derivatives (like $[x'(t)]^2$) are allowed.

On a computer, we might solve something like this with brute force:

1. start the mass at $x = 2$ and $\frac{dx}{dt} = 0$ (i.e. initial conditions are a stretched spring that is not moving.)

2. use $\frac{d^2x}{dt^2} = \frac{-k}{m} x$ to figure out accelleration.

3. use $\frac{dx}{dt} \approx \left(\frac{\partial^2 x}{dt^2}\right)_{t=0} t_1 = v(t_1)$ to find velocity at t_1

4. use $x \cdot \left(\frac{dx}{dt}\right)_{t=t_1} (t_2 - t_1)$ to find $x(t_2)$

Now the position has changed so go back to step 2.

This procedure generates a trajectory, once we know $x(0)$ & $v(0)$ we can predict at any future time: $x(t)$ & $v(t)$
You need knowledge of the previous location or the most you can learn is: if the mass is here, next

time it will be there.

We can and will solve exactly this differential equation (not by brute force).

In general DEs can be set up easily and are much more difficult to solve, but they can capture some very complex and important behavior, for example, we could add friction:

$$F = F_{spring} + F_{drag}$$

$$= -kx - \gamma \frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0$$

Linear First-Order Equations

Radioactive decay: In a sample of radioactive material any given atom has a random chance of decaying in a given time.

An atom of ^{210}Po has a 0.021% chance of decaying in an hour's time, and that is independent of the chemical environment.

Decay of a sample of ^{210}Po is therefore dependent on how much we have at any time:

$$\frac{d[Po]}{dt} = -k[Po]$$

What is $[Po](t)$? i.e what is the function that describes the time dependence of Po concentration?

One way to solve this is to <u>guess</u> a function that matches the Diff eq. This equation says the 1^{st} derivative of the funct must be equal to the function (scaled by $-k$)

Here's a function:

$$[Po] = e^{-kt}$$

$$\frac{d[Po]}{dt} = -k[Po]$$

$$\frac{d}{dt}(e^{-kt}) = -k(e^{-kt})$$

$$-ke^{-kt} = -ke^{-kt} \quad \checkmark$$

We could also have guessed $2e^{-kt}$ or $\frac{1}{3}e^{-kt}$, so the most general solution will include a constant in front:

$$[Po] = Ae^{-kt}$$

A has physical meaning. At $t=0$, $[Po] = Ae^{0} = A$ so it is the <u>initial</u> amount present in the sample.

Guessing is a poor choice for many DE, so we need something more useful:

Split the differential operator: $\frac{d[P_o]}{dt} = -k[P_o]$

$$\frac{d[P_o]}{[P_o]} = -k\,dt$$

Integrate both sides:

$$\int \frac{1}{[P_o]}\, d[P_o] = -\int k\,dt$$

$$\ln[P_o] = -kt + C$$

$$e^{\ln[P_o]} = e^{-kt} e^{C}$$

$$[P_o] = e^{-kt} e^{C} = Ae^{-kt}$$

This works for more complicated 1^{st} order equations also:

$$\frac{dy}{dx} + p(x)\,y = 0 \qquad (p(x) \text{ is a function of } x)$$

$$\frac{dy}{dx} = -p(x)\,y$$

$$\frac{dy}{y} = -p(x)\,dx$$

$$\int \frac{dy}{y} = -\int p(x)\,dx$$

$$\ln y + C = -\int p(x)\,dx$$

$$y = Ae^{-\int p(x)\,dx}$$

Now, what about:

$$\frac{dy}{dx} + p(x)y = q(x)$$

This trick is often used

- If $q(x)=0$, then we'd have our previous solution

$$y = Ae^{-\int p(x)\,dx}$$

- We guess that the solution will "look like" this and try

$$y = u(x)\,e^{-\int p(x)\,dx}$$

- Plug this back into the DE to find out what we can about $u(x)$:

Remember:
$$\frac{d}{dx} e^{f(x)} = \frac{df}{dx} e^{f(x)}$$

So:
$$\frac{d}{dx}\left(Ae^{-\int p(x)\,dx}\right) = \frac{d}{dx}\left(-\int p(x)\,dx\right)\left[Ae^{-\int p(x)\,dx}\right)$$

$$= -A\,p(x)e^{-\int p(x)\,dx}$$

The product rule gives us:

$$\frac{d}{dx}\left(u(x)\,e^{-\int p(x)\,dx}\right) = u(x)\frac{d}{dx}\left(e^{-\int p(x)\,dx}\right) + \frac{du}{dx}e^{-\int p(x)\,dx}$$

$$= -u(x)p(x)\,e^{-\int p(x)\,dx} + \frac{du}{dx}\,e^{-\int p(x)\,dx}$$

Now we plug this into our DE:

$$-u(x)\,p(x)\,e^{-\int p(x)\,dx} + \frac{du}{dx}e^{-\int p(x)\,dx} + p(x)u(x)e^{-\int p(x)\,dx} = q(x)$$

$$\therefore \frac{du}{dx}\,e^{-\int p(x)\,dx} = q(x)$$

$$du = q(x)e^{+\int p(x)dx}\,dx$$

$$\int du = \int q(x)\, e^{+\int p(x)\,dx}\,dx$$

$$u(x) = \int q(x)\, e^{\int p(x)\,dx}\,dx + C$$

$$\therefore \quad y(x) = e^{-\int p(x)dx}\left(\int q(x)e^{+\int p(x)dx}\,dx + C\right)$$

Here's a totally arbitrary example.

$$x^2\frac{dy}{dx} + xy = 3$$

We squeeze it into this form:

$$\frac{dy}{dx} + p(x)\,y = q(x)$$

$$\frac{dy}{dx} + \frac{y}{x} = \frac{3}{x^2} \qquad p(x) = \frac{1}{x}, \quad q(x) = \frac{3}{x^2}$$

$$\int p(x)\,dx = \int \frac{1}{x}\,dx = \ln x$$

$$e^{-\int p(x)dx} = e^{-\ln x} = \frac{1}{x}$$

$$\therefore \quad y(x) = \frac{1}{x}\left(\int x\, q(x)\,dx + C\right)$$

$$= \frac{C}{x} + \frac{1}{x}\int \frac{3}{x}\,dx$$

$$\boxed{y(x) = \frac{C}{x} + \frac{3\ln x}{x}}$$

Here's an example from a 2-step reaction:

$$A \xrightarrow{k_1} B \xrightarrow{k_2} C$$

This gives differential equations:

$$\frac{dA}{dt} = -k_1 A$$

← A disappears with a rate that exactly mirrors the Polonium example.

$$\frac{dB}{dt} = k_1 A - k_2 B$$

$$\frac{dC}{dt} = k_2 B$$

If, at time $t=0$, ~~the~~ we have only $A = A_0$, $B=0$, $C=0$ in our sample, then:

initial conditions

$$A(t) = A_0 e^{-k_1 t}$$

2^{nd} equation is trickier:

$$\frac{dB}{dt} = k_1 A - k_2 B$$

$$\underbrace{\frac{dB}{dt} + k_2 B}_{p(t)B} = \underbrace{k_1 A_0 e^{-k_1 t}}_{q(t)} \implies p(t) = k_2 \quad q(t) = k_1 A_0 e^{-k_1 t}$$

$$y(t) = e^{-\int p(t)dt}\left(\int q(t) e^{+\int p(t)dt} dt + \beta\right)$$

$$B(t) = e^{-k_2 t}\left(\int k_1 A_0 e^{-k_1 t} e^{k_2 t} dt + \beta\right)$$

$$= e^{-k_2 t}\left(k_1 A_0 \int e^{(k_2 - k_1)t} dt + \beta\right)$$

$$= e^{-k_2 t}\left(\frac{k_1 A_0}{k_2 - k_1} e^{(k_2 - k_1)t} + \beta\right)$$

$$B(t) = \frac{k_1 A_0}{k_2 - k_1} e^{-k_1 t} + \beta e^{-k_2 t}$$

Now, we can use the initial conditions: $B(0) = 0$

$$0 = \frac{k_1 A_0}{k_2 - k_1} e^{0} + \beta e^{0}$$

$$\beta = \frac{-k_1 A_0}{k_2 - k_1}$$

$$B(t) = \frac{k_1 A_0}{k_2 - k_1} \left(e^{-k_1 t} - e^{-k_2 t} \right)$$

We could go ahead and do $\frac{dC}{dt}$, but it is easier to use conservation of mass:

$$A(t) + B(t) + C(t) = A(0) + B(0) + C(0) = A_0$$

$$C(t) = A_0 - A_0 e^{-k_1 t} - \frac{k_1 A_0}{k_2 - k_1} \left(e^{-k_1 t} - e^{-k_2 t} \right)$$

This mechanism helps explain the following chain:

$$^{232}Th \xrightarrow{t_{1/2} = 10^{10} y} {}^{220}Rn \xrightarrow{t_{1/2} = 60 s} {}^{208}Pb$$

Rate is inversely proportional to half-life:

$$k = \frac{\ln 2}{t_{1/2}}$$

If the sample of thorium is sealed, so the Radon gas can't escape, we reach a quasi steady-state of Radon by assuming:

$$k_1 \approx 0$$

$$k_2 \gg \frac{1}{t}$$

Then:

$$A(t) = A_0 e^{-k_1 t} \approx A_0$$

$$k_1 A(t) \approx k_1 A_0$$

$$B(t) = \frac{k_1 A_0}{k_2 - k_1}\left(e^{-k_1 t} - e^{-k_2 t}\right)$$

$$\approx \frac{k_1 A_0}{k_2}$$

$$k_2 B(t) \approx k_1 A_0$$

Higher order equations.

A second order equation gets written in a general way as:

$$a_2(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)\, y = f(x)$$

With higher derivatives, we have

$$\sum_{i=0}^{\infty} a_i(x) \frac{d^i y}{dx^i} = f(x)$$

In general, these equations are not solvable analytically if they contain nonlinear terms:

$$y^2 \quad \text{or} \quad y \frac{dy}{dx}$$

And they are <u>hard</u> unless $f(x) = 0$

When $f(x)$ is zero, the equations are called <u>homogeneous</u>.

To get to these, we'll introduce <u>operator</u> notation:

an operator acting on a function

$$\mathcal{L}\, y(x) = 0$$

In the case of a 2^{nd} order homogeneous equation $\mathcal{L}$ could be quite complex:

$$\mathcal{L} = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)$$

$$\mathcal{L}\; y(x) = \left[a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)\right] y(x)$$

$$= a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y(x)$$

Multiplication and differentiation are linear & distributive, so

$$a_0(x)\cdot c\,y(x) = c\,a_0(x)\,y(x)$$

$$a_0(x)\left[y_1(x) + y_2(x)\right] = a_0(x)y_1(x) + a_0(x)y_2(x)$$

$$\frac{d}{dx}\left[c_1 y_1(x) + c_2 y_2(x)\right] = c_1\frac{dy_1}{dx} + c_2\frac{dy_2}{dx}$$

We can combine solutions to the diff eq using these properties:

If $y_1(x)$ & $y_2(x)$ are both solutions to the differential equation

$$\mathcal{L}y = 0$$

then the combinations $c_1 y_1(x) + c_2 y_2(x)$ is also a solution.

$$\mathcal{L}y_1 = 0 \qquad \mathcal{L}y_2 = 0$$

$$\mathcal{L}[c_1 y_1 + c_2 y_2] = c_1\mathcal{L}y_1 + c_2\mathcal{L}y_2$$

$$= c_1\cdot 0 + c_2\cdot 0$$

$$= 0$$

This means that any number of solutions can be added together with any weightings and they'll still be a solution.

2nd order Homogeneous Equations

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0$$

We'll try a guess $y = e^{\alpha x}$ first

$$\frac{d^2}{dx^2} e^{\alpha x} - 2\frac{d}{dx} e^{\alpha x} - 3e^{\alpha x} = 0$$

$$\alpha^2 e^{\alpha x} - 2\alpha e^{\alpha x} - 3e^{\alpha x} = 0$$

$$e^{\alpha x}(\alpha^2 - 2\alpha - 3) = 0$$

Because $e^{\alpha x}$ is never $= 0$, we need:

$$\alpha^2 - 2\alpha - 3 = 0$$

$$(\alpha - 3)(\alpha + 1) = 0$$

or $\alpha = 3$ & $\alpha = -1$

This tells us that our general solutions are:

$$y_1 = e^{-x} \qquad y_2 = e^{3x}$$

Let's check:

$$\frac{d^2}{dx^2} e^{-x} - 2\frac{d}{dx} e^{-x} - 3e^{-x} = 0$$

$$+e^{-x} + 2e^{-x} - 3e^{-x} = 0$$

$$0 = 0 \quad ✓$$

Likewise $\frac{d^2}{dx^2} e^{3x} - 2\frac{d}{dx} e^{3x} - 3e^{3x} = 0$

$$9e^{3x} - 6e^{3x} - 3e^{3x} = 0 \quad ✓$$

Also, combinations of these solutions

$$y(x) = c_1 e^{-x} + c_2 e^{3x}$$ ← 2 constants that must be determined by boundary conditions

Here's another trickier example:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Follow same steps: $y = e^{\alpha x}$

$$\alpha^2 e^{\alpha x} - 4\alpha e^{\alpha x} + 4e^{\alpha x} = 0$$

$$\alpha^2 - 4\alpha + 4 = 0$$

$$(\alpha - 2)^2 = 0$$

We've got a double root $\alpha = 2$ $\quad y(x) = e^{2x}$

But 2^{nd} order equations always have at least 2 distinct solutions, so here's a trick:

$$y(x) = \underbrace{u(x)}_{\text{just like the } p(x)-q(x) \text{ method}} e^{2x}$$

$$\therefore \quad \frac{d^2}{dx^2}\left[u(x)e^{2x}\right] - 4\frac{d}{dx}\left[u(x)e^{2x}\right] + 4\left[u(x)e^{2x}\right] = 0$$

$$\frac{d}{dx}\left[e^{2x}\frac{du}{dx} + 2u(x)e^{2x}\right] - 4\left[e^{2x}\frac{du}{dx} + 2u(x)e^{2x}\right] + 4u(x)e^{2x} = 0$$

$$2e^{2x}\frac{du}{dx} + e^{2x}\frac{d^2u}{dx^2} + 4u(x)e^{2x} + 2e^{2x}\frac{du}{dx} - 4e^{2x}\frac{du}{dx} - 8u(x)e^{2x} + 4u(x)e^{2x} = 0$$

$$e^{2x}\left[\frac{d^2u}{dx^2} + (2+2-4)\frac{du}{dx} + (4u - 8u + 4u)\right] = 0$$

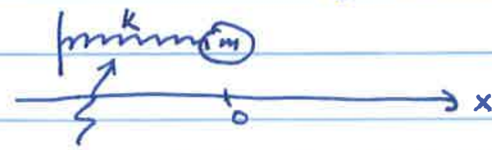
$$\frac{d^2u}{dx^2} = 0$$

$$u(x) = c_1 x + c_2$$

So, the final answer is:

$$y(x) = c_1 x e^{2x} + c_2 e^{2x}$$

Now consider what differential equation we might get for a mass on a spring



$$F = ma = m\frac{d^2x}{dt^2} \qquad F = -kx \leftarrow \text{Hooke's law}$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

k and m are both positive numbers so let's temporarily replace them with an easy number, like 4.

$$\frac{d^2x}{dt^2} + 4x = 0 \qquad \leftarrow \text{2nd order, homogeneous, linear.}$$

$$x(t) = e^{\alpha t} \qquad \leftarrow \text{try this}$$

$$\alpha^2 e^{\alpha t} + 4e^{\alpha t} = 0$$

$$\underbrace{e^{\alpha t}}_{\text{Never } 0}\underbrace{(\alpha^2 + 4)}_{} = 0$$

$$\alpha^2 + 4 = 0$$

$$\alpha^2 = -4$$

$$\alpha = \pm\sqrt{-4} = \pm 2i$$

This means:

$$x(t) = c_1 e^{+2it} + c_2 e^{-2it}$$

Does this make sense?

Let's use Euler's relation to simplify this a bit:

$$x(t) = c_1(\cos 2t + i\sin 2t) + c_2(\cos 2t - i\sin 2t)$$

$$= \underbrace{(c_1+c_2)}_{C_3} \cos 2t + \underbrace{i(c_1-c_2)}_{C_4} \sin 2t$$

$$x(t) = C_3 \cos 2t + C_4 \sin 2t \quad \leftarrow \text{general solution}$$

The solutions are oscillatory. The position cycles back & forth symmetrically around $x=0$.

Boundary conditions are things we know at a specific value of the independent variable. For example, we might know the mass starts at position x_0 with a velocity of 0 at $t=0$.

$$x(0) = x_0 \quad , \quad \left.\frac{dx}{dt}\right|_{t=0} = 0$$

Since:

$$x(t) = C_3 \cos 2t + c_4 \sin 2t$$

$$x(0) = C_3 \cos(0) + c_4 \sin(0) = C_3 = x_0$$

$$\frac{dx}{dt} = -2c_3 \sin 2t + 2c_4 \cos 2t$$

$$\left.\frac{dx}{dt}\right|_{t=0} = -2c_3 \sin(0) + 2c_4 \cos(0) = 2c_4 = 0$$

$\therefore$ $C_3 = x_0$, $c_4 = 0$ and

$$x(t) = x_0 \cos 2t \quad \leftarrow \text{Specific solution}$$

You might be curious about the complex form above:

$$x(t) = c_1 e^{2it} + c_2 e^{-2it}$$

$$x(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = x_0$$

$$x'(t) = 2c_1 e^{2it} - 2c_2 e^{-2it}$$

$$x'(0) = 2c_1 e^0 - 2c_2 e^0 = 2c_1 - 2c_2 = 0$$

$$\therefore \quad c_1 = c_2$$

$$\therefore \quad 2c_1 = x_0 \quad \text{or} \quad c_1 = \frac{x_0}{2}, \quad c_2 = \frac{x_0}{2}$$

$$x(t) = \frac{x_0}{2} e^{2it} + \frac{x_0}{2} e^{-2it} = \frac{x_0}{2}\left[e^{2it} + e^{-2it}\right]$$

$$x(t) = x_0 \cos(2t) \longleftarrow$$ same result once we apply the initial boundary conditions

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Damped Oscillator : This has an additional frictional term proportional to velocity which puts on the brakes when the oscillator moves quickly:

$$m \frac{d^2x}{dt^2} = \underbrace{-kx}_{\text{spring force}} \underbrace{- m\gamma \frac{dx}{dt}}_{\text{frictional drag}}$$

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0 \longleftarrow \omega_0 = \sqrt{\frac{k}{m}}$$

Plug in $x(t) = e^{\alpha t}$ to get a polynomial

$$\alpha^2 e^{\alpha t} + \gamma \alpha e^{\alpha t} + \omega_0^2 e^{\alpha t} = 0$$

$$\underbrace{e^{\alpha t}}_{\text{never } 0,\ \text{so:}}(\alpha^2 + \gamma\alpha + \omega_0^2) = 0$$

$$(\alpha^2 + \gamma\alpha + \omega_0^2) = 0$$

$$\alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

Whether or not this oscillates depends on the stuff inside the $\sqrt{\ }$

If $\gamma^2 > 4\omega_0^2$, $\sqrt{\gamma^2 - 4\omega_0^2} > 0$ and both values of α are real

Call $\xi^2 = \gamma^2 - 4\omega_0^2$, then $\alpha = \frac{-\gamma}{2} \pm \frac{\xi}{2}$ and the solutions decay in time

If $\gamma^2 < 4\omega_0^2$ then the $\sqrt{\ }$ is complex

use

$\chi = 4\omega_0^2 - \gamma^2$ and our α values are

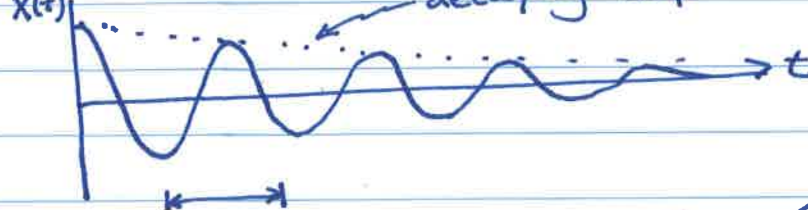
$$\alpha = -\frac{\gamma}{2} \pm \frac{i\chi}{2}$$

The general solution looks like:

$$\begin{aligned}
x(t) &= e^{\alpha t} \\
&= c_1 e^{-(\gamma + i\chi)t/2} + c_2 e^{-(\gamma - i\chi)t/2} \\
&= c_1 e^{-t\gamma/2} e^{i\chi t/2} + c_2 e^{-t\gamma/2} e^{-i\chi t/2} \\
&= e^{-t\gamma/2}\left[c_1 e^{i\chi t/2} + c_2 e^{-i\chi t/2}\right] \\
&= e^{-\gamma t/2}\left[c_3 \cos\left(\frac{\chi t}{2}\right) + c_4 \sin\left(\frac{\chi t}{2}\right)\right]
\end{aligned}$$

This is a <u>decaying</u> oscillation:

decaying amplitude ($e^{-\gamma t/2}$)



oscillatory behavior ($\cos\frac{\chi t}{2} + \sin\frac{\chi t}{2}$)

For the special case where

$$\gamma^2 = 4\omega_0^2$$

the spring is said to be <u>critically damped</u> and $x(t)$ returns to 0 as fast as possible.

Many automatic door closers are critically damped springs!