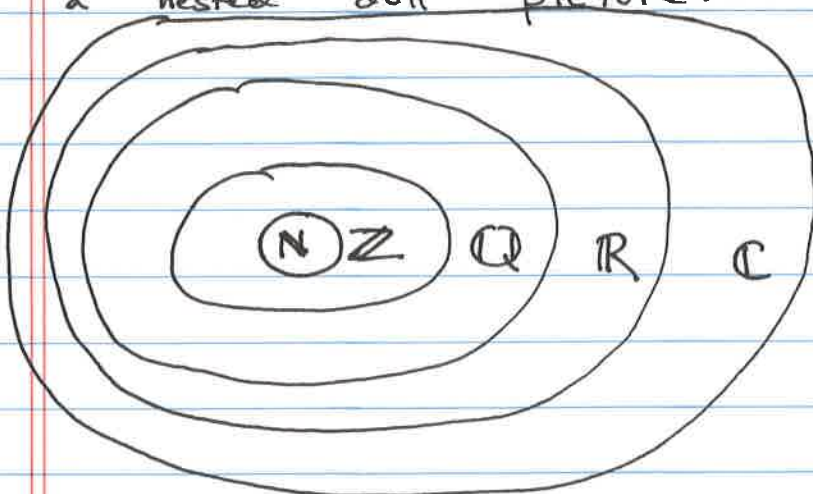
Numbers, Complex Numbers, Polynomials

We're going to approach the concept of numbers with a nested doll picture:



And we'll think about the kinds of math operations we can do with each set of numbers

The innermost (and smallest set) of numbers is the natural numbers,

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

The natural numbers are useful abstractions of things we can count, 3 rocks, 3 sheep, 3 trees, and give us the most important concept of math the abstraction from the concrete object. we talk about "threeness" as a separate concept and now think of 3 as a separate object. (A similar level of abstraction occurs in chemistry with concepts like electron, orbital, bond, etc.)

With $\mathbb{N}$, we can add things together and multiply, and our result will always be part of the same group of numbers, i.e.

$$2+3=5 \quad \text{in } \mathbb{N} \quad 2*3=6$$

$\mathbb{N}$ is <u>closed</u> under addition & multiplication because these operations on any 2 members of $\mathbb{N}$ returns another member of $\mathbb{N}$

$\mathbb{N}$ is <u>not</u> closed under <u>subtraction</u> or <u>division</u>

$12 - 17$ = something <u>not</u> in $\mathbb{N}$

$12/5$ = something <u>not</u> in $\mathbb{N}$

The <u>subtraction</u> problem was solved by the discovery (in India) around 1400 years ago of zero, and by the discovery in Europe (during the Renaissance) of the set of Integers, $\mathbb{Z}$ ($\mathbb{Z}$ for Zahl, german for number)

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

$\mathbb{Z}$ is <u>closed</u> under addition, subtraction, multiplication

Division is <u>still</u> a problem, however, and to get a system of numbers that is closed under division, we need the Rational numbers $\mathbb{Q}$ (for quotient)

$\mathbb{Q}$ are <u>dense</u>, that is between any 2 members of $\mathbb{Q}$, there is always another one!

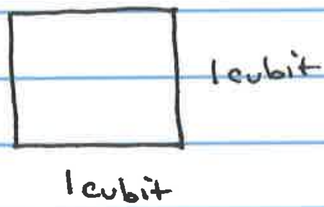
$$\underbrace{\frac{1190507}{10292881} \qquad \frac{185015}{1599602}}_{\text{differ by only 1 part in 16 trillion}}$$

$$\frac{2300597}{19890493}$$

sits between the other 2

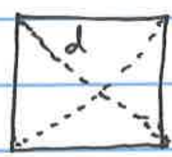
$\mathbb{Q}$ includes $\mathbb{Z}$ which includes $\mathbb{N}$
(i.e. all integers are honorary rational numbers)
so there are <u>many</u> more rationals than integers

A problem in greek & egyptian architecture



Suppose we want to build a perfectly square building with walls 1 cubit on a side.

To make sure it is square, we can stretch 2 identical ropes across the <u>diagonals</u> and make sure they meet in the center.



How long are the ropes?

$$d^2 = 1^2 + 1^2 = 2$$
$$d = \sqrt{2}$$

It is pretty easy to show that although $\mathbb{Q}$ is closed under addition, subtraction, mult & division, the square root produces numbers that <u>cannot</u> be represented by rational numbers. They are non-terminating & non-repeating

$\mathbb{R}$ ~~are~~ is the set of <u>real</u> numbers.
Is $\mathbb{R}$ <u>closed</u> under $\sqrt{\ }$?

-1 is in $\mathbb{R}$, but is $\sqrt{-1}$?

No! To have closed numbering system under all operations, we need one more set, the complex numbers, $\mathbb{C}$.

Complex numbers are built around 2 copies of the real numbers

$$c = a + bi \qquad a, b \in \mathbb{R}$$

i is the $\sqrt{-1}$ and is the fundamental imaginary quantity.

All imaginary numbers are complex (with $a=0$)
All real numbers are complex (with $b=0$)

a & b are the real & imaginary parts of a complex number:

$$Re(c) = a$$
$$Im(c) = b$$

Arithmetic with complex numbers is usually easy, but requires some bookkeeping to keep track of your i values:

$$c_1 = a_1 + b_1 i$$
$$c_2 = a_2 + b_2 i$$

$$c_1 + c_2 = (a_1 + a_2) + (b_1 + b_2) i$$

Subtraction is also straightforward:

$$c_1 - c_2 = (a_1 - a_2) + (b_1 - b_2)i$$

Multiplying by a <u>real</u> involves distributing

$$c = a + bi$$
$$kc = k(a + bi)$$
$$= ka + kbi$$

Multiplying two complex numbers involves just a bit more care:

$$c_1 \times c_2 = (a_1 + b_1 i) \times (a_2 + b_2 i)$$
$$= a_1 a_2 + a_1 b_2 i + b_1 i a_2 + b_1 i b_2 i$$
$$= a_1 a_2 + i(a_1 b_2 + b_1 a_2) + b_1 b_2 \underbrace{i^2}_{= -1}$$
$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$$

A <u>trick</u> to divide: $\dfrac{c_1}{c_2} = \dfrac{a_1 + b_1 i}{a_2 + b_2 i} \cdot \boxed{\dfrac{(a_2 - b_2 i)}{(a_2 - b_2 i)} \quad \text{Multiply by } 1}$

$$\frac{c_1}{c_2} = \frac{(a_1 a_2 + b_1 b_2) + i(b_1 a_2 - a_1 b_2)}{a_2^2 - a_2 b_2 i + a_2 b_2 i - b_2^2 i^2}$$

$$= \frac{(a_1 a_2 + b_1 b_2) + i(b_1 a_2 - a_1 b_2)}{a_2^2 + b_2^2} \longleftarrow \text{Denominator is } \underline{\text{Real}}, \text{ so:}$$

$$\frac{c_1}{c_2} = \underbrace{\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}}_{Re\left[\frac{c_1}{c_2}\right]} + i \underbrace{\frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}}_{Im\left[\frac{c_1}{c_2}\right]}$$

The Complex Plane & Complex Conjugation

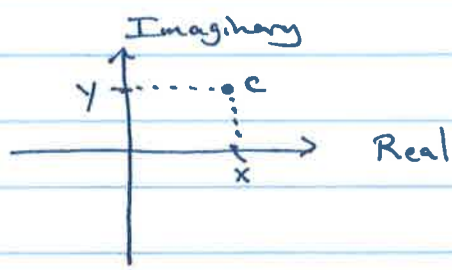
Last time, we talked about complex numbers in terms of a <u>real</u> part and an imaginary part:

$$c = x + yi$$

If we have a complex number, we can figure out both x & y and likewise if we know x & y, this uniquely determines c.

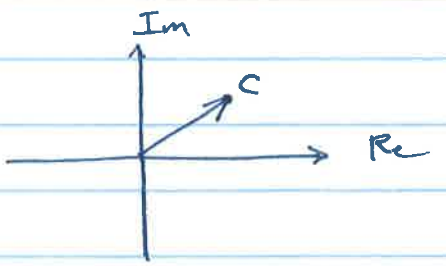
$$c \longleftrightarrow (x, y)$$

We can think of x & y as coordinates on a cartesian coordinate system

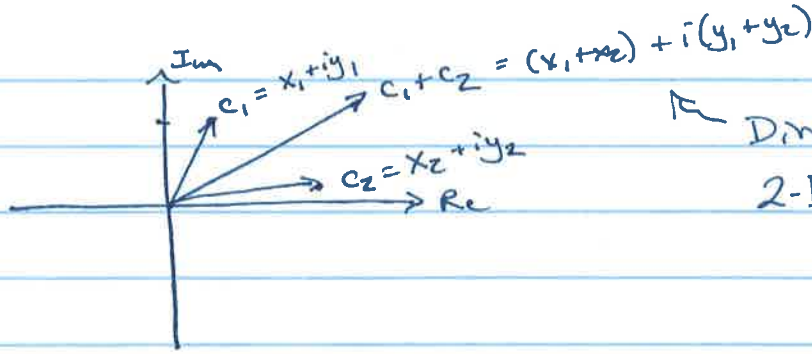


By convention, we draw the real axis as horizontal. Every complex number is a <u>point</u> on this <u>complex plane</u>.

We can also think of a complex number as a vector

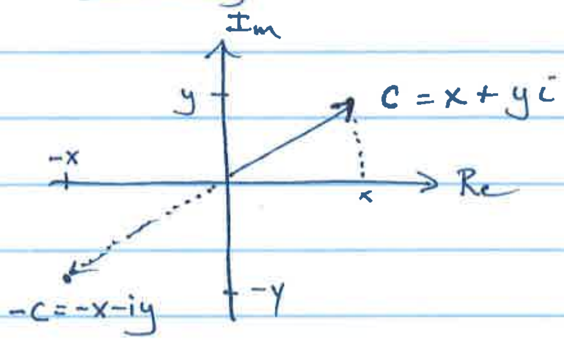


Addition of 2 complex numbers is analogous to adding 2 vectors (head-to-tail):

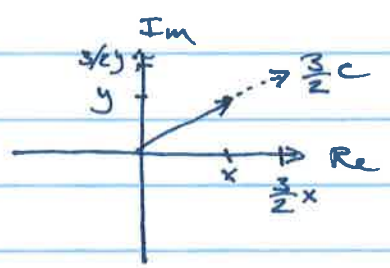


Directly analogous to 2-D vector addition.

The negative of a complex number is just like reflecting the vector through the origin:

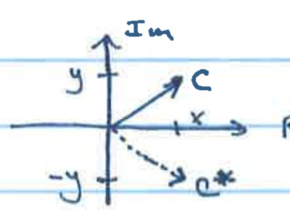


Multiplying a complex number by a real (scalar) is like scaling the length of the vector:



$$\frac{3}{2} c = \frac{3}{2} x + \frac{3}{2} iy$$

A particularly interesting operation is called complex conjugation or reflection across the x-axis:



This operation takes

$$x + yi \longrightarrow x - yi$$

and we denote it with an asterisk:

$$c^* = (x + yi)^* = x - yi$$

A few things to note about <u>complex conjugation</u>

- The complex conjugate of a <u>real</u> is the same number

$$x^* = (x + 0i)^* = x - 0i = x$$

- The complex conjugate of a complex conjugate returns the original number

$$(c^*)^* = ((x+yi)^*)^* = (x-yi)^* = x+yi$$
$$= c$$

On your homework, you'll derive:

$$\mathrm{Re}[c] = \frac{1}{2}(c + c^*)$$

$$\mathrm{Im}[c] = \frac{1}{2i}(c - c^*)$$

$$= \frac{1}{2i}\left((x+yi) - (x-yi)\right)$$

$$= \frac{1}{2i}\left(x - x + yi + yi\right)$$

$$= \frac{1}{2i}(2yi) = y$$

The real utility comes from multiplying a complex number by its <u>own</u> conjugate:

$$cc^* = (x+yi)(x-yi)$$
$$= x^2 - x(iy) + iy(x) - i^2y^2$$
$$= x^2 - (-1)y^2$$
$$= x^2 + y^2 \leftarrow$$

we define this operation as the <u>Square</u> <u>Modulus</u>

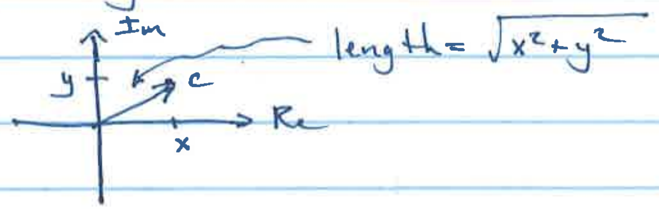
$$|c|^2 = cc^* = x^2 + y^2$$

If x & y are Real (and they are) then the square modulus is always a positive real number, and is equal to 0 only when $c = 0$

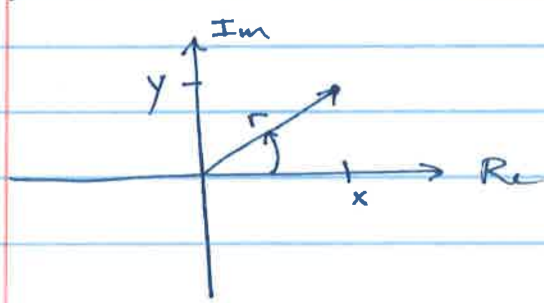
The modulus or absolute value of a complex number is:

$$|c| = \sqrt{|c|^2} = \sqrt{x^2+y^2}$$

Geometrically, $|c|$ is the length of the vector representing c on the complex plane:



Representing complex numbers in polar form



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = |c| = \sqrt{x^2+y^2}$$

$$c = r\cos\theta + i\,r\sin\theta$$
$$c = r(\cos\theta + i\sin\theta)$$

$$\theta = \tan^{-1}\frac{y}{x} = \mathrm{Arg}[c] = \text{argument of } c$$

3 forms:

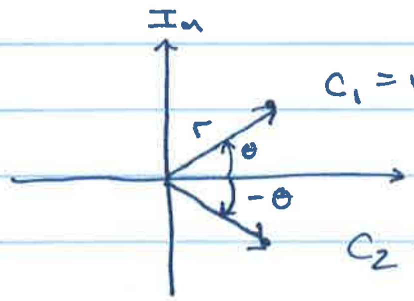
$$c = (x + iy) = r(\cos\theta + i\sin\theta)$$

So we can go back & forth between:

$$c(x,y) \longleftrightarrow c(r,\theta)$$

The polar form is usually best for <u>multiplication</u> & <u>powers</u>, while the cartesian form works best for addition & subtraction

Consider a flip from $+\theta \longrightarrow -\theta$



$$c_1 = r[\cos\theta + i\sin\theta]$$

$$c_2 = r[\cos(-\theta) + i\sin(-\theta)]$$

$$= r[\cos\theta - i\sin\theta]$$

$$= c_1^* \leftarrow$$ complex conjugation just changes the <u>sign</u> of the <u>angle</u>, θ.

In general,

$$c_1 c_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 \big[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)\big]$$

$$c_1 c_2 = r_1 r_2 \big[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big]$$

product — multiply the <u>moduli</u> — <u>add</u> the angles

Euler's Relation

Here's a Taylor series refresher:

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \ldots$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \ldots$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 \ldots$$

We can do a little creative rewriting of these to handle complex numbers:

Remember $i^2 = -1$

$i^3 = -i$

$i^4 = +1$

$$\cos x = 1 + \frac{1}{2}i^2x^2 + \frac{1}{4!}i^4x^4 + \frac{1}{6!}i^6x^6 \ldots$$

$$\sin x = x + \frac{1}{3!}i^2x^3 + \frac{1}{5!}i^4x^5 + \frac{1}{7!}i^6x^7$$

Let's multiply both sides by i:

$$i\sin x = ix + \frac{1}{3!}i^3x^3 + \frac{1}{5!}i^5x^5 + \frac{1}{7!}i^7x^7$$

If we put these two together we get:

$$\cos x + i\sin x = \underbrace{1 + ix + \frac{1}{2}(ix)^2 + \frac{1}{3!}(ix)^3 + \frac{1}{4!}(ix)^4 + \frac{1}{5!}(ix)^5 + \ldots}_{\text{looks exactly like the Taylor series for } e^x \text{ (but with } ix)}$$

This is Euler's relation: $\boxed{e^{ix} = \cos x + i\sin x}$

Written in terms of an angle:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

(This leads to one of the most amazing equations ever discovered:

$$e^{i\pi} = -1 + 0$$

$$\boxed{e^{i\pi} - 1 = 0}$$

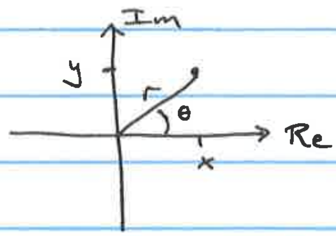
← contains all 5 of the most fundamental numbers in mathematics!

More on Euler's relation

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

one of the most useful equations!

We now know three ways to represent a complex number



$$c = x + iy$$ Cartesian form

$$\updownarrow$$

$$c = r\left[\cos\theta + i\sin\theta\right]$$ polar form

$$\updownarrow$$

$$c = re^{i\theta}$$ Euler form

These are all inter connected:

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right] \longleftrightarrow \left[\begin{array}{l} r = |c| = \sqrt{x^2+y^2} \qquad \text{(modulus of c)} \\ \tan\theta = \frac{y}{x} \quad \Rightarrow \quad \theta = \arg(c) = \tan^{-1}\frac{y}{x} \end{array}\right.$$

The Cartesian form is best for addition/subtraction, while polar form or Euler form are usually best for multiplication, powers, etc.

$$c_1 = r_1 e^{i\theta_1} \qquad c_2 = r_2 e^{i\theta_2}$$

$$c_1 c_2 = r_1 r_2 \, e^{i(\theta_1+\theta_2)} = r_1 r_2 \left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

$$c_1 = r_1 e^{i\theta_1} \longrightarrow \sqrt{c_1} = c_1^{1/2}$$

$$= \left[r_1 e^{i\theta_1}\right]^{1/2}$$

$$= \sqrt{r_1}\, e^{i\theta_1/2}$$

This is also useful to deal with complex conjugates

$$c^* = (x+iy)^* = x - iy$$
$$= r\cos\theta - ir\sin\theta \quad = r[\cos\theta - i\sin\theta]$$

Remember: $\cos(-x) = \cos(x)$ ← cos is symmetric

$\sin(-x) = -\sin(x)$ ← sin is anti-symmetric

So:

$$c^* = r[\cos(-\theta) + i\sin(-\theta)]$$
$$= re^{-i\theta}$$

So the square modulus is easy in Euler form

$$c^*c = (re^{-i\theta})(re^{i\theta}) = r^2 e^{i(\theta-\theta)} = r^2$$

We can use Euler's relation to simplify expressions involving sin or cos:

$$e^{i\theta} + e^{-i\theta} = (\cos\theta + i\sin\theta) + (\cos(-\theta) + i\sin(-\theta))$$
$$= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$
$$= 2\cos\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{likewise:} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

This feature allows us to simplify some very difficult integrals, like this one, common in NMR Spectroscopy:

$$\int_0^\infty \sin t \, e^{-\alpha t} \, dt = \int_0^\infty \left[\frac{1}{2i}\left(e^{it} - e^{-it}\right)\right] e^{-\alpha t} \, dt$$

$$= \frac{1}{2i} \int_0^\infty \left(e^{it-\alpha t} - e^{-it-\alpha t}\right) dt$$

$$= \frac{1}{2i} \left[\int_0^\infty e^{(i-\alpha)t} \, dt - \int_0^\infty e^{(-i-\alpha)t} \, dt \right]$$

$$= \frac{1}{2i} \left[\left[\frac{e^{-(\alpha-i)t}}{-(\alpha-i)} \right]_0^\infty - \left[\frac{e^{-(\alpha+i)t}}{-(\alpha+i)} \right]_0^\infty \right]$$

$$= \frac{1}{2i} \left[\left[0 - \frac{1}{-(\alpha-i)} \right] - \left[0 - \frac{1}{-(\alpha+i)} \right] \right]$$

$$= \frac{1}{2i} \left[\frac{1}{\alpha-i} - \frac{1}{\alpha+i} \right]$$

$$= \frac{1}{2i} \left[\frac{1}{\alpha-i} \frac{(\alpha+i)}{(\alpha+i)} - \frac{1}{\alpha+i} \frac{(\alpha-i)}{(\alpha-i)} \right]$$

$$= \frac{1}{2i} \left[\frac{\alpha+i}{\alpha^2-i^2} - \frac{\alpha-i}{\alpha^2-i^2} \right] = \frac{1}{2i} \left(\frac{2i}{\alpha^2+1} \right)$$

$$= \frac{1}{\alpha^2+1}$$

<u>Functions of complex variables</u>

$w = f(z)$ ← w & z are complex numbers

$z = x + iy$ ← cartesian form for z

$w = u(x,y) + i\, v(x,y)$ ← cartesian form for w

u & v are functions that map

$z \longrightarrow w$

Let's do an <u>example</u>:

$$w = f(z) = z^2$$

$$w = (x+iy)(x+iy)$$
$$= \underbrace{x^2-y^2}_{} + i\,\underbrace{2xy}_{}$$

we identify imaginary part as $v(x,y)$

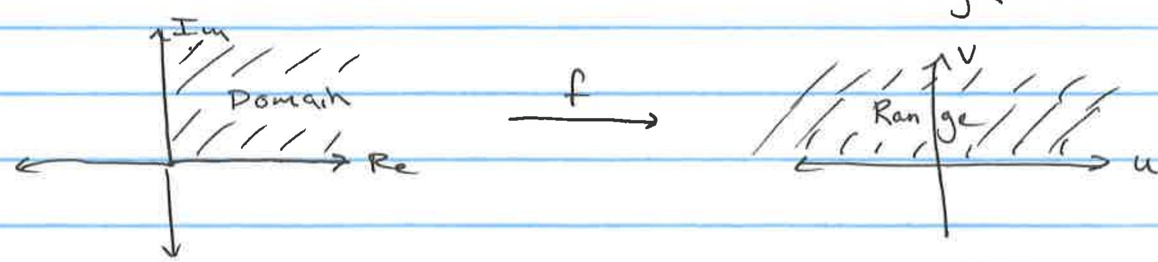
we identify <u>real</u> part as $u(x,y)$

$$\therefore \quad u(x,y) = x^2 - y^2$$
$$v(x,y) = 2xy$$

<u>Domain</u> : a set of "valid" points for z in complex plane

<u>Range</u> : a set of "valid" points for w in complex plane

<u>Example</u>: $w = f(z) = z^2$ for $0 \le x \le \infty$, $0 \le y < \infty$



$$-\infty < u < \infty$$
$$0 \le v < \infty$$

<u>de Moivre's Formula</u> (an application of Euler):

$$z^n = r^n (\cos\theta + i\sin\theta)^n$$
$$= r^n e^{in\theta}$$
$$= r^n \left[\cos(n\theta) + i\sin(n\theta)\right]$$

<u>Example</u>: Let $n=2$ & $r=1$

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$$

By de Moivre's formula, however

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

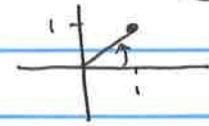
Comparing Real & imaginary parts, we get:

$$\left.\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\cos\theta\sin\theta \end{aligned}\right]$$

some useful identities

de Moivre's formula is particularly useful for powers:

$z = 1+i$



$\rightarrow \quad z = \sqrt{2}\, e^{i\pi/4}$

So:

$$(1+i)^3 = \left(\sqrt{2}\, e^{i\pi/4}\right)^3 = 2^{3/2} e^{i3\pi/4}$$

$$= 2^{3/2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$$

Hyperbolic Functions

Remember: $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

For complex variables, we can invent similar functions

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

Which have an identity

$$\cosh^2 z - \sinh^2 z = \frac{(e^z + e^{-z})^2}{4} - \frac{(e^z - e^{-z})^2}{4}$$

$$= \frac{e^{2z} + e^{-2z} + 2 - e^{2z} - e^{-2z} + 2}{4}$$

$$= 1$$

Hyperbolic & regular functions are related

$$\cos(iz) = \cosh(z)$$
$$\sin(iz) = i\sinh(z)$$

$$\cosh(iz) = \cos(z)$$
$$\sinh(iz) = i\sin(z)$$

This is useful: to calculate sins & cos for complex numbers:

$$\sin(z) = \sin(x+iy)$$
$$= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$
$$= \frac{e^{ix-y} - e^{-ix+y}}{2i}$$
$$= \frac{1}{2i}\left[e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right]$$
$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$