Ising Model Partition Functions

nearest neighbors

$$\mathcal{H} = -\frac{J}{2} \sum_{n} \sum_{n'}{}' \sigma_n \sigma_{n'}$$

In 1-D: (no field, no periodic bounderies)

$$\mathcal{H} = -\frac{J}{2}\left(\underbrace{\sigma_1\sigma_2 + (\sigma_2\sigma_1}_{} + \underbrace{\sigma_2\sigma_3) + (\sigma_3\sigma_2}_{} + \underbrace{\sigma_3\sigma_4)}_{} + \cdots\right.$$

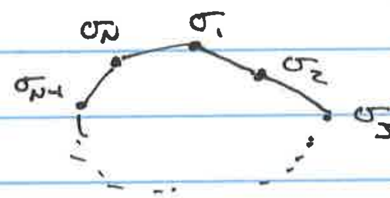
We can recombine these terms together

$$\mathcal{H} = -J \sum_{n=1}^{N-1} \sigma_n \sigma_{n+1}$$

written so each spin only couples to the next one in line

With Periodic Bounderies:

(edge effects are gone because $\sigma_{N+1} = \sigma_1$)



$$\mathcal{H} = -J \sum_{n=1}^{N} \sigma_n \sigma_{n+1}$$

one more term

We can define a <u>bond</u> variable $b_i = \sigma_i \sigma_{i+1}$

σ_i	σ_{i+1}	b_i
+1	+1	+1
+1	−1	−1
−1	+1	−1
−1	−1	+1

we'll need an additional factor of 2 for each lattice to account for degenerate states!

For N spins, we need N−1 bond variables (and a factor of 2) to visit all states

$$Q_N = \sum_{\{\sigma_i = \pm 1\}}^{N} e^{\beta J \sum_i \sigma_i \sigma_{i+1}} = 2 \sum_{\{b_i = \pm 1\}}^{N-1} e^{\beta J \sum_i b_i}$$

$$Q_N = 2 \sum_{\{b_i = \pm 1\}} e^{\beta J b_1} e^{\beta J b_2} e^{\beta J b_3} \cdots e^{\beta J b_{N-1}}$$

$$= 2\left(\sum_{b_1=\pm 1} e^{\beta J b_1}\right)\left(\sum_{b_2=\pm 1} e^{\beta J b_2}\right) \cdots \left(\sum_{b_{N-1}} e^{\beta J b_{N-1}}\right)$$

$$= 2\left(e^{\beta J} + e^{-\beta J}\right)\left(e^{\beta J} + e^{-\beta J}\right) \cdots \left(e^{\beta J} + e^{-\beta J}\right)$$

$$\boxed{Q_N = 2\left(2\cosh \beta J\right)^{N-1}}$$

Next, without periodic boundaries or dual-lattice bond variables:

$$\mathcal{H} = -J \sum_{n=1}^{N-1} \sigma_n \sigma_{n+1}$$

$$Q_N = \sum_{\{\sigma_i = \pm 1\}}' e^{\beta J \sum_{n=1}^{N-1} \sigma_n \sigma_{n+1}}$$

$$= \sum_{\{\sigma_i = \pm 1\}}' e^{\beta J \sigma_1 \sigma_2} e^{\beta J \sigma_2 \sigma_3} \cdots e^{\beta J \sigma_{N-2} \sigma_{N-1}} e^{\beta J \sigma_{N-1} \sigma_N}$$

Let's do the last spin first ($\sigma_N = \pm 1$):

$$Q_N = \sum_{\{\sigma_i = \pm 1\}}' e^{\beta J \sigma_1 \sigma_2} \cdots e^{\beta J \sigma_{N-2}\sigma_{N-1}} \underbrace{\left(e^{\beta J \sigma_{N-1}} + e^{-\beta J \sigma_{N-1}}\right)}_{\parallel}$$

If $\sigma_{N-1} = +1$: $\left(e^{\beta J} + e^{-\beta J}\right)$

If $\sigma_{N-1} = -1$: $\left(e^{-\beta J} + e^{\beta J}\right)$

$$\parallel$$

$\therefore$ $2\cosh \beta J$

$$Q_N = 2\cosh \beta J \sum_{\{\sigma_i = \pm 1\}}' e^{\beta J \sum_{n=1}^{N-2} \sigma_n \sigma_{n+1}} = (2\cosh \beta J)\, Q_{N-1}$$

We can continue this sequence all the way down to Q_1:

$$Q_N = (2\cosh\beta J)^{N-1}\, Q_1$$

$$= (2\cosh\beta J)^{N-1} \sum_{\sigma_1=\pm 1}^{1} \sum_{\sigma_2=\pm 1}^{1} e^{\beta J \sigma_1 \sigma_2}$$

These last terms we can do explicitly:

$$Q_N = (2\cosh\beta J)^{N-1} \Big(\underbrace{e^{\beta J}}_{\substack{\sigma_1=1\\ \sigma_2=1}} + \underbrace{e^{-\beta J}}_{\substack{\sigma_1=1\\ \sigma_2=-1}} + \underbrace{e^{-\beta J}}_{\substack{\sigma_1=-1\\ \sigma_2=1}} + \underbrace{e^{\beta J}}_{\substack{\sigma_1=-1\\ \sigma_2=-1}} \Big)$$

$$Q_N = (2\cosh\beta J)^{N-1}\; 2\left(e^{\beta J} + e^{-\beta J}\right)$$

$$\boxed{Q_N = 2\cdot(2\cosh\beta J)^N}$$ ← without periodic boundaries

$$\boxed{Q_N = 2\cdot(2\cosh\beta J)^{N-1}}$$ ← with periodic boundaries

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Free Energies, Heat Capacities

$$A(N,V,T) = -k_B T \ln Q_N \quad = -k_B T\left[\ln 2 + N\ln(2\cosh\beta J)\right]$$

$$= -kT\ln 2 - NkT\ln\left[2\cosh\beta J\right]$$

$$\langle E\rangle = +kT^2\left(\frac{\partial \ln Q}{\partial T}\right) = -\left(\frac{\partial \ln Q}{\partial \beta}\right) = -N\frac{1}{\cancel{2}\cosh\beta J}\cdot \cancel{2}\sinh\beta J\cdot J$$

$$= -NJ\tanh\beta J$$

$$C_V = \frac{\partial\langle E\rangle}{\partial T} = \frac{+J^2 N}{kT^2}\left(\operatorname{sech}\frac{J}{k_B T}\right)^2$$ ← why might this be a problem?

With a field

$$\mathcal{H} = -H\sum_{i=1}^{N}\sigma_i - J\sum_{i=1}^{N-1}\sigma_i\sigma_{i+1} \xrightarrow{\text{PBC}} = -\frac{H}{2}\sum_{i=1}^{N}(\sigma_i+\sigma_{i+1}) - J\sum_{i=1}^{N}\sigma_i\sigma_{i+1}$$

$$Q = \sum_{\{\sigma_i=\pm1\}} e^{\beta J\sum_i \sigma_i\sigma_{i+1} + \frac{\beta H}{2}\sum_i(\sigma_i+\sigma_{i+1})}$$

Let's define a <u>transfer</u> <u>matrix</u> $\underline{\underline{P}}$:

it connects adjacent spins σ & σ'

$$\langle\sigma|P|\sigma'\rangle = e^{\beta J\sigma\sigma' + \beta H(\sigma+\sigma')/2}$$

$$\langle+1|P|+1\rangle = e^{\beta(J+H)}$$
$$\langle+1|P|-1\rangle = e^{-\beta J}$$
$$\langle-1|P|+1\rangle = e^{-\beta J}$$
$$\langle-1|P|-1\rangle = e^{\beta(J-H)}$$

$$\underline{\underline{P}} = \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-H)} \end{pmatrix}$$ ← connects <u>states</u> of two adjacent spins

$$Q = \sum_{\{\sigma_i=\pm1\}} \langle\sigma_1|P|\sigma_2\rangle\langle\sigma_2|P|\sigma_3\rangle\cdots\langle\sigma_N|P|\sigma_1\rangle$$

↳ this includes <u>all</u> possible states:

$$\sum_{\sigma_1=\pm1}\sum_{\sigma_2=\pm1}\cdots\sum_{\sigma_N=\pm1}$$

If we use the closure relation: $\sum_{\sigma_i=\pm1}|\sigma_i\rangle\langle\sigma_i| = 1$

$$Q = \sum_{\sigma_1=\pm1}\langle\sigma_1|\underline{\underline{P}}^N|\sigma_1\rangle = \mathrm{Tr}\left[\underline{\underline{P}}^N\right]$$

To carry out the trace, we must first diagonalize $\underline{\underline{P}}$

A brief interlude on 2×2 matrices

$$\underline{\underline{A}} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \underline{\underline{B}} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$[A \cdot B]_{ij} = \sum_{k=1}^{2} A_{ik} B_{kj}$$

$$\mathrm{Tr}[\underline{\underline{A}}] = A_{11} + A_{22} = \sum_{k=1}^{2} A_{kk}$$

The <u>trace</u> is conserved for <u>cyclic</u> permutations

$$\mathrm{tr}[A \cdot B \cdot C] = \mathrm{tr}[C \cdot A \cdot B] = \mathrm{tr}[B \cdot C \cdot A]$$

but <u>not</u> for acyclic permutations:

$$\mathrm{tr}[A \cdot B \cdot C] \neq \mathrm{tr}[B \cdot A \cdot C]$$

Diagonalization

$$\underline{\underline{M}} = \underline{\underline{U}}^T \cdot \underline{\underline{A}} \cdot \underline{\underline{U}}$$

For an arbitrary square matrix $\underline{\underline{A}}$, there is a <u>unitary</u> transformation which results in a <u>diagonal</u> matrix $\underline{\underline{M}}$

$$\underline{\underline{M}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \longleftarrow \lambda_1 \text{ \& } \lambda_2 \text{ are the } \underline{\text{eigenvalues}} \text{ of } \underline{\underline{A}}$$

$\underline{\underline{U}}$ = matrix of unit <u>eigenvectors</u> of $\underline{\underline{A}}$

columns of $\underline{\underline{U}}$ are eigenvectors of $\underline{\underline{A}}$

$$\left.\begin{aligned} \underline{\underline{A}} \cdot \vec{u}_1 &= \lambda_1 \vec{u}_1 = \lambda_1 \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} \\ \underline{\underline{A}} \cdot \vec{u}_2 &= \lambda_2 \vec{u}_2 = \lambda_2 \begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix} \end{aligned}\right\} \longrightarrow \underline{\underline{U}} = \begin{pmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{pmatrix}$$

The diagonalization transform is a <u>unitary</u> similarity transform:

$$\underline{\underline{U}}^T = \underline{\underline{U}}^{-1}$$

$$\underline{\underline{U}}^T \cdot \underline{\underline{U}} = \underline{\underline{U}}^{-1} \cdot \underline{\underline{U}} = \underline{\underline{I}}$$ The Identity Matrix

Now, consider: $\mathrm{Tr}[\underline{\underline{P}}^N] = \sum_k [P^N]_{kk}$ hard to determine

Suppose we diagonalize $\underline{\underline{P}}$ first:

$$\underline{\underline{M}} = \underline{\underline{U}}^T \cdot \underline{\underline{P}} \cdot \underline{\underline{U}}$$

$$\underline{\underline{M}}^N = (\underline{\underline{U}}^T \cdot \underline{\underline{P}} \cdot \underline{\underline{U}})(\underline{\underline{U}}^T \cdot \underline{\underline{P}} \cdot \underline{\underline{U}})(\underline{\underline{U}}^T \cdot \underline{\underline{P}} \cdot \underline{\underline{U}}) \cdots$$

$$= U^T \cdot P \cdot \underbrace{(\underline{\underline{U}} \cdot \underline{\underline{U}}^T)}_{U^{-1} = U^T} \cdot P \cdot (\underline{\underline{U}} \cdot \underline{\underline{U}}^T) \cdot P \cdots$$

$$= \underline{\underline{U}}^T \cdot \underline{\underline{P}} \cdot \underline{\underline{I}} \cdot \underline{\underline{P}} \cdot \underline{\underline{I}} \cdot \underline{\underline{P}} \cdots$$

$$\underline{\underline{M}}^N = \underline{\underline{U}}^T \cdot \underline{\underline{P}}^N \cdot \underline{\underline{U}}$$

So:

$$\mathrm{Tr}[\underline{\underline{M}}^N] = \mathrm{Tr}[\underline{\underline{U}}^T \cdot \underline{\underline{P}}^N \cdot \underline{\underline{U}}]$$ cyclic permutation

$$= \mathrm{Tr}[\underline{\underline{U}} \cdot \underline{\underline{U}}^T \cdot \underline{\underline{P}}^N]$$

$$\mathrm{Tr}[\underline{\underline{M}}^N] = \mathrm{Tr}[\underline{\underline{P}}^N]$$

$$\therefore Q_N = \mathrm{tr}\left[\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^N\right] = \mathrm{tr}\left[\begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix}\right] = \lambda_1^N + \lambda_2^N$$

Now, back to the Ising problem:

$$\mathcal{H} = \sum_n \left[-J\sigma_n\sigma_{n+1} - \frac{H}{2}(\sigma_n + \sigma_{n+1}) \right]$$

$$Q_N = \sum_{\sigma_1=\pm1} \cdots \sum_{\sigma_N=\pm1} \langle \sigma_1 | \underbrace{e^{\beta J\sigma_1\sigma_2 + \frac{\beta H}{2}(\sigma_1+\sigma_2)}} | \sigma_2 \rangle \langle \sigma_2 | \cdots$$

$\underline{\underline{P}}$ = transfer matrix connecting σ_1 & σ_2

$$\parallel$$

$$\begin{pmatrix} e^{\beta J + \beta H} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta H} \end{pmatrix}$$

$$Q_N = \mathrm{Tr}[\underline{\underline{P}}^N] = \mathrm{Tr}[\underline{\underline{U}}^T \cdot \underline{\underline{M}}^N \cdot \underline{\underline{U}}] = \mathrm{Tr}[\underline{\underline{M}}^N]$$

$$= M_{11}^N + M_{22}^N = \lambda_1^N + \lambda_2^N$$ $\leftarrow$ λ_1 & λ_2 are eigenvalues of $\underline{\underline{P}}$

$$\det[\underline{\underline{P}} - \lambda \underline{\underline{I}}] = 0 \Rightarrow \begin{vmatrix} e^{\beta J + \beta H} - \lambda & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta H} - \lambda \end{vmatrix} = 0$$

$$(e^{\beta J + \beta H} - \lambda)(e^{\beta J - \beta H} - \lambda) - e^{-2\beta J} = 0$$

$$e^{2\beta J} - \lambda(e^{\beta J + \beta H} + e^{\beta J - \beta H}) + \lambda^2 - e^{-2\beta J} = 0$$

$$\lambda^2 - \lambda(e^{\beta J}(e^{\beta H} + e^{-\beta H})) + (e^{2\beta J} + e^{-2\beta J}) = 0$$

$$\lambda^2 - \lambda e^{\beta J}(2\cosh\beta H) + 2\sinh(2\beta J)$$

$$\lambda = \frac{e^{\beta J}2\cosh\beta H \pm \sqrt{e^{2\beta J}4\cosh^2\beta H - 8\sinh 2\beta J}}{2}$$

$$= e^{\beta J}\cosh\beta H \pm \sqrt{e^{2\beta J}\cosh^2\beta H - 2\sinh(2\beta J)}$$

$$= e^{\beta J}\cosh\beta H \pm \sqrt{e^{2\beta J}\cosh^2\beta H - e^{2\beta J} + e^{-2\beta J}}$$

$$= e^{\beta J}\left(\cosh\beta H \pm \sqrt{\cosh^2\beta H - 1 + e^{-4\beta J}}\right)$$

$$\boxed{\lambda_\pm = e^{\beta J}\left(\cosh\beta H \pm \sqrt{\sinh^2\beta H + e^{-4\beta J}}\right)}$$

$$Q_N = \lambda_+^N + \lambda_-^N \longleftarrow$$

one will always be larger

$1.1^N + 0.9^N$

$\Uparrow$ will dominate as $N \to \infty$

$$Q_N \approx \left(e^{\beta J}\left(\cosh\beta H + \sqrt{\sinh^2\beta H + e^{-4\beta J}}\right)\right)^N$$

$$A \approx -Nk_BT\ln\left[e^{\beta J}\cosh\beta H + \left(e^{2\beta J}\sinh^2\beta H + e^{-2\beta J}\right)^{1/2}\right]$$

$$m = \langle\sigma\rangle = -\frac{1}{N}\frac{\partial A}{\partial H} = \frac{1}{\beta\lambda_+}\frac{\partial\lambda_+}{\partial H}$$

$$m = \frac{\sinh(\beta H)}{\sqrt{\sinh^2 \beta H + e^{-\beta J \cdot 4}}}$$

$\therefore$ when $H \to 0$ there is <u>no</u> spontaneous magnetization at <u>any</u> temperature in 1-D.

In 2D, there is!

An experimental tie:

magnetic susceptibility : $\chi = \frac{\partial \langle m \rangle}{\partial H}$

A review of what we know:

$$\mathcal{H} = -H\sum_{i=1}^{N}\sigma_i - \frac{J}{2}\sum_{i=1}^{N}\sum_{j\in NN_i}\sigma_i\sigma_j$$

nearest neighbor sum

$0^\circ K$ states with $H=0$;

$J>0 \longrightarrow$ degenerate ferromagnetic states
all up, $\langle m\rangle = +1$
all down, $\langle m\rangle = -1$

$J<0 \longrightarrow$ degenerate anti-ferromagnetic states
$+-+-$ and $-+-+$
both with $\langle m\rangle = 0$

At any temperature in 1D, we've shown:

$$Q_N = 2(2\cosh\beta J)^N \quad \leftarrow \text{no field}$$

$$Q_N = \left(e^{\beta J}\left(\cosh\beta H + \sqrt{\sinh^2\beta H - e^{-4\beta J}}\right)\right)^N \quad \leftarrow \text{field}$$

we got here using a transfer matrix, diagonalization, and the cyclic invariance of the trace

Here are some derivative tricks:

$$\langle m\rangle = \frac{1}{Q}\sum_{\{\sigma_i=\pm1\}}{}^{'}\frac{1}{N}\left(\sum_{i=1}^{N}\sigma_i\right)e^{\beta\left(H\sum_i\sigma_i + \sum_{ij}\sigma_i\sigma_j\cdot\frac{J}{2}\right)}$$

this term in exponent matches

$$\langle m\rangle = \frac{1}{N}\frac{\partial\ln Q}{\partial(\beta H)} = \frac{1}{N}\cdot\frac{1}{Q}\frac{\partial Q}{\partial(\beta H)}$$

$$= \frac{1}{N}\frac{1}{Q}\frac{\partial}{\partial(\beta H)}\sum_{\{\}}e^{\beta H\sum_i\sigma_i + \frac{\beta J}{2}\sum_i\sum_j\sigma_i\sigma_j}$$

this derivative pulls down this term

$$= \frac{1}{Q} \sum_{\{\}} \frac{1}{N}\left(\sum_i \sigma_i\right) e^{\beta H \sum_i \sigma_i + \frac{\beta J}{2} \sum_i \sum_j' \sigma_i \sigma_j}$$

$$\therefore \quad \langle m \rangle = \frac{1}{N} \frac{\partial \ln Q}{\partial (\beta H)} = \frac{k_B T}{N} \frac{\partial \ln Q}{\partial H}$$

Last time, we also showed:

$$\langle m \rangle = \frac{\sinh(\beta H)}{\sqrt{\sinh^2 \beta H + e^{-4\beta J}}}$$

The other first derivative property of interest

$$\frac{\langle E \rangle}{N} = \frac{1}{N} - \frac{\partial \ln Q}{\partial \beta} = -J \tanh(\beta J)$$

has no discontinuities

The second derivative properties

Susceptibility

$$\chi = \frac{\partial m}{\partial H}$$

$$= \frac{\beta \cosh(\beta H)}{\left(1 + e^{4\beta J} \sinh^2(\beta H)\right)^{3/2}} + \text{other ugly terms}$$

$$\lim_{H \to 0} \chi = \frac{\beta}{\sqrt{e^{-4\beta J}}} \leftarrow \text{only diverges at } T=0$$

Heat Capacity

$$C_V = \frac{\partial \langle E \rangle}{\partial T}$$

$$= \frac{J}{kT^2} \operatorname{sech}^2(\beta J)$$

no divergences!

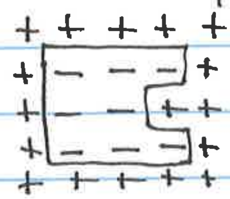
Conclusions: There are no phase transitions in the 1-D Ising model!

Peierls Theorem

For the 2D Ising model, there exists a temperature T_c at which the probability of "+" spins $\neq$ the probability of "−" spins

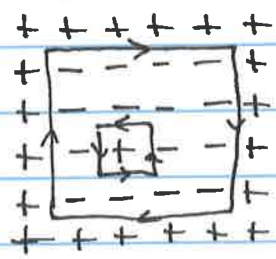
That is, $\langle \sigma_n \rangle \neq 0$ below T_c

Consider an array of spins:



← energy = J × length of perimeter

N spins on an array, with all edge spins set to "+"



a contour passes through the midpoint of every + − bond

a closed contour meets itself

$$C(\ell, i)$$

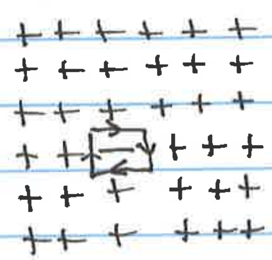
ℓ ← length

i ← index or label

Energy of a closed contour $\quad E[C] = J\ell$

Direction : R.H.S. of path has "−" spins

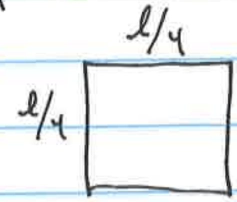
Conjugate : Reverse all spins to RHS of contour



$$\tilde{C}(\ell, i)$$

$$E[\tilde{C}] = E[C] - \ell J$$

The contour with the maximal number of enclosed spins for a given length is a regular polygon:



$$A = N_{max} = \frac{l^2}{16}$$

$\mathcal{M}(l)$ = total number of contours of length l

$$\mathcal{M}(l) \leq \text{total number of contours we can draw}$$

$$\leq \underbrace{N}_{\text{places to start}} \times \underbrace{4}_{\text{choices for } 1^{st} \text{ step}} \times \underbrace{3^{l-1}}_{\text{choices of direction on all following steps}} \times \frac{1}{2l} \longleftarrow \text{required to close loop}$$

$$X(l,i) = \begin{cases} 1 & \text{if configuration contains contour } C(l,i) \\ 0 & \text{otherwise} \end{cases}$$

$$\underbrace{N_-}_{\#\text{ of negative spins}} \leq \sum_l \underbrace{\left(\frac{l^2}{16}\right)}_{\text{maximal enclosed spins}} \sum_{i=1}^{\mathcal{M}(l) \,\leftarrow\, \text{maximal \# of contours of length } l} \underbrace{X(l,i)}_{\text{does configuration contain this contour}} = \underline{\text{overestimate}} \text{ of } N_-$$

$$\langle X(l,i) \rangle = \frac{\sum_{\{\sigma_n\}} e^{-\beta \mathcal{H}(\{\sigma_n\})} X(l,i)}{\sum_{\{\sigma_n\}} e^{-\beta \mathcal{H}(\{\sigma_n\})}}$$

we can do this as a "constrained" sum over only those configurations containing $C(l,i)$

$$\langle X(\ell,i)\rangle = \frac{\sum'_{\substack{\text{constrained}\\ \text{configurations}}} e^{-\beta E[c(\ell,i)]}}{\sum_{\substack{\text{all}\\ \text{states}}} e^{-\beta \mathcal{H}}}$$

← we can underestimate the denominator as the same set of constrained set of configurations without the contour

$$\langle X(\ell,i)\rangle \leq \frac{\sum'_{\text{configs}} e^{-\beta E[c(\ell,i)]}}{\sum_{\text{configs}} e^{-\beta E[\tilde{c}(\ell,i)]}}$$

$$\langle x(\ell,i)\rangle \leq e^{-\beta J \ell}$$ ← since $E[\tilde{c}] = E[c] - J\ell$

So:

$$\frac{\langle N_-\rangle}{N} \leq \frac{1}{N} \sum_{\ell} \left(\frac{\ell^2}{16}\right) \sum_{i=1}^{m(\ell)} \langle x(\ell,i)\rangle$$

$$\leq \sum_{\ell=4}^{\infty} \left(\frac{\ell^2}{16}\right) 3^{\ell-1} e^{-J\beta\ell} \frac{4}{2\ell}$$

$$\frac{\langle N_-\rangle}{N} \leq \frac{\frac{5}{24} e^{-4\alpha}}{1-e^{-\alpha}}$$ where $\alpha = \frac{J}{T} - \ln 3$

$$\frac{\langle N_-\rangle}{N} \ll \frac{1}{2} \quad \text{as} \quad T \to 0$$

$\therefore$ There must be a T_c below which $\frac{\langle N_-\rangle}{N} < \frac{1}{2}$

2-D exact solution:

Lars Onsager Phys. Rev. 65, pp. 117-149 (1944).

$$Q(\beta, N) = \left[2\cosh(\beta J\; e^{I}\right]^N$$

$$I = \frac{1}{2\pi}\int_0^{\pi} d\phi \,\ln\left\{\frac{1}{2}\left[1 + (1-\kappa^2\sin^2\phi)^{1/2}\right]\right\}$$

$$\kappa = \frac{2\sinh\beta J}{\cosh^2(2\beta J)}$$

Critical Temperature: $T_c = \dfrac{2.269\, J}{k_B}$

$$\frac{C_V}{N} \sim \frac{8k_B}{\pi}(\beta J)^2 \ln\left|\frac{1}{T-T_c}\right| \qquad \alpha = 0$$

$$m \sim \text{const}\,(T_c - T)^{1/8} \quad \text{when} \quad T < T_c$$

$$\beta = 1/8$$

3D: No exact solutions yet! Numerically:

$$\frac{C_V}{N} \propto |T-T_c|^{-0.125}$$

$$m \propto (T_c - T)^{0.313} \qquad T < T_c$$

$$T_c \sim \frac{4J}{k_B}$$