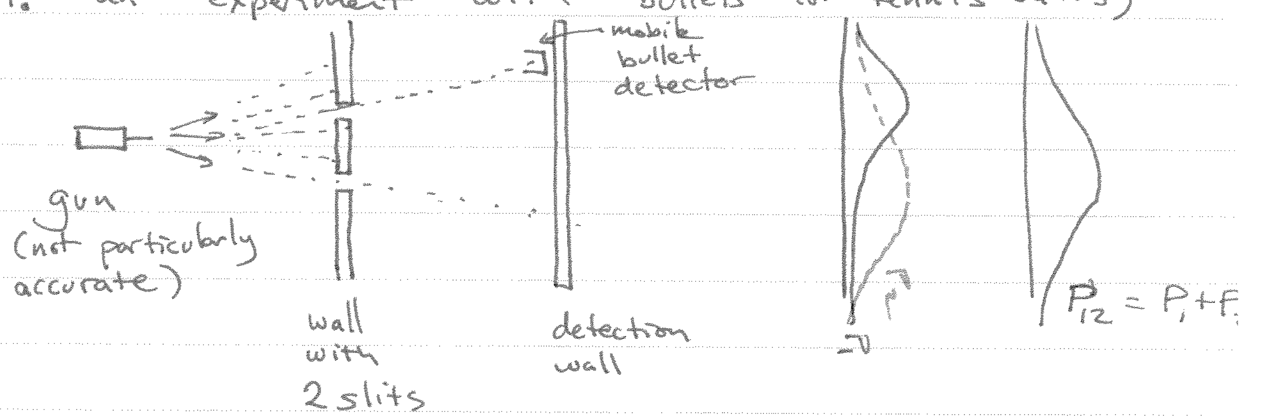
What is Quantum Mechanics?

1. The most successful (yet!) scientific description of how matter & energy interact on an atomic scale. This makes it the <u>foundation</u> of modern chemistry.

2. A frustrating mix of thought experiments, postulates, methods, and bizarre conclusions. "If you think you understand Quantum Mechanics, you probably don't."

Let's start with the <u>essential Quantum Paradox</u>:
2-slit experiments

1. an experiment with bullets (or tennis balls)



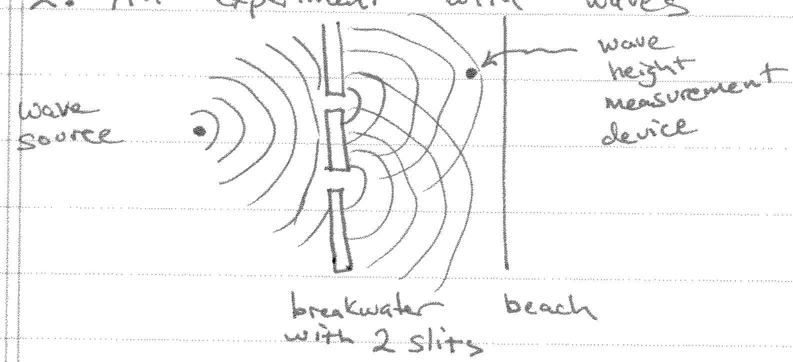
- The detector can be "counted" i.e. we can stop the experiment and count the number of bullets in the detector.
- A bullet may "bounce" off the edge of a slit and go anywhere.
- The amount of time the detector spends at each location is the same.

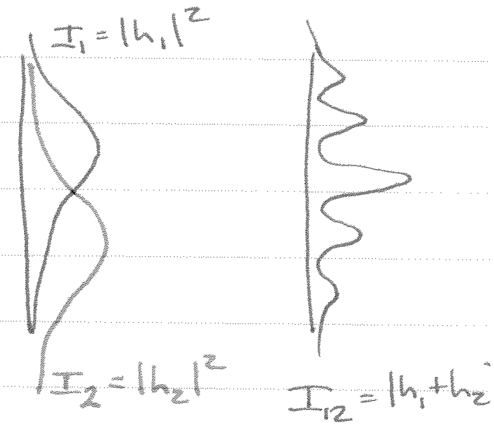
- Bullets arrive one at a time, but after a very long experiment, we obtain a distribution of bullets that came through each slit.

- The distribution when <u>both</u> slits are open is P_{12}. This is the <u>sum</u> of the distributions we get when only 1 slit is open.

$$P_{12} = P_1 + P_2$$

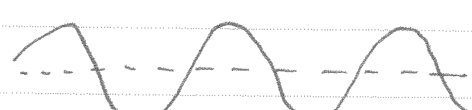
2. An experiment with waves



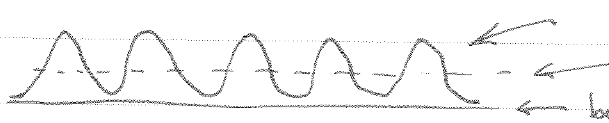


- water waves are generated by a buoy that has a motor driving it up and down so that it makes circular waves

- The detector measures the intensity $= (\text{height})^2$ of waves that impinge on the beach



quiescent level

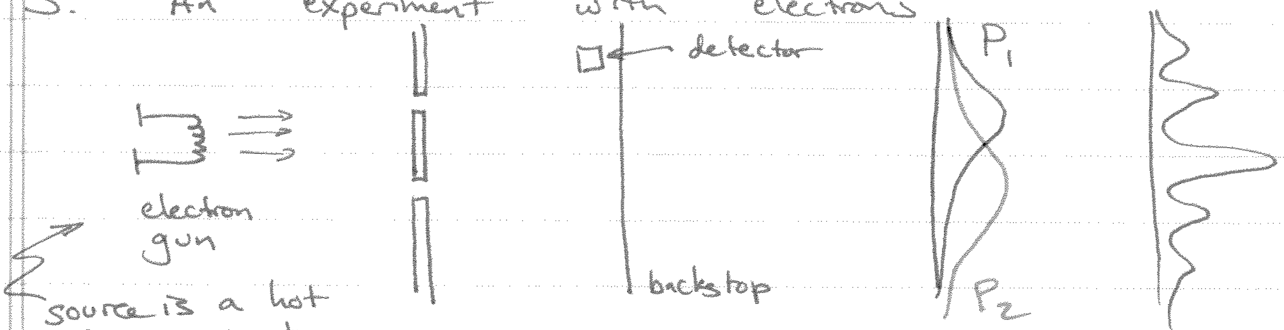


$|h|^2 =$ average energy imparted to beach

bottom

- An interference pattern develops because the 2 slits cause new circular wave patterns to form

3. An experiment with electrons



source is a hot wire kept at a voltage below the backstop

detector is a geiger counter or PMT connected to a speaker

$P_{12} \neq P_1 + P_2$

- There are no "partial" electrons. They arrive as particles (just like bullets). We can verify this with multiple detectors and very low discharge rates.

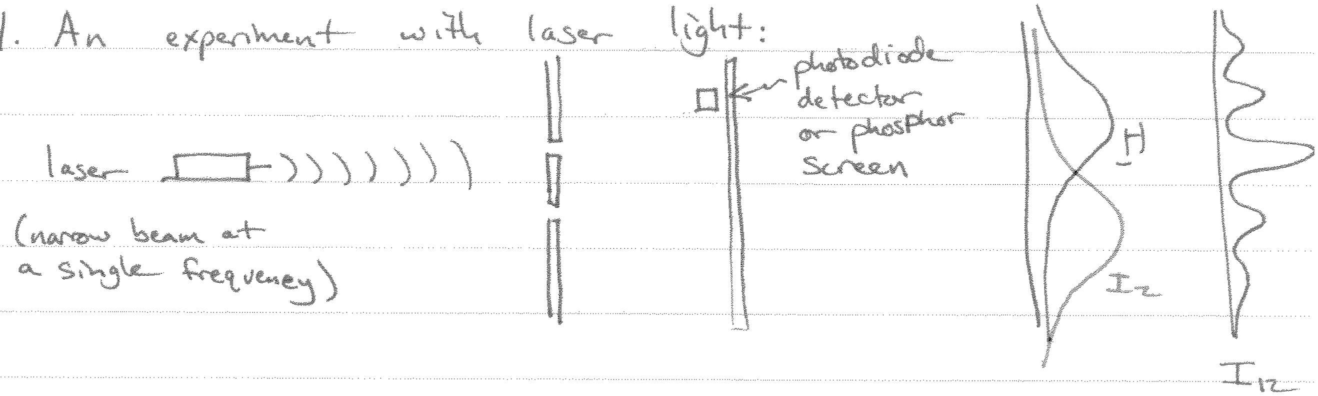
- When we move the detector, we can measure the intensity at a location from the average number of clicks/second: More electrons = more clicks

Proposition: Let's say that each electron goes through either hole 1 or hole 2.

We can check this by blocking hole 1 and hole 2 (one at a time), then do the experiment again with both open. The proposition would require $P_{12} = P_1 + P_2$

So electrons are going through both slits simultaneously and collapsing to a single position as they hit the detector.

4. An experiment with laser light:



What will happen if we lower the fluence of the laser so much that only a few photons are hitting the slits each second?

Do we get individual particle-like clicks at the detector?

Math Review

- You should know basic integration & differentiation
- I'll teach you some of the other techniques we'll use: (Differential Equations, Linear Algebra, integration tricks)

Quantum Mechanics uses complex numbers which have both real & imaginary parts:

$$z = a + \underbrace{bi}$$

complex ← z, real ← a, imaginary ← bi

$$i = \sqrt{-1} \Rightarrow i^2 = -1$$

These numbers come about naturally from the roots of quadratic equations:

$$z^2 - 2z + 5 = 0$$

$$z = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm \sqrt{-4}$$

$$z = 1 \pm 2i$$

For this particular number: $Re[z] = 1$

$Im[z] = \pm 2$

In general if $z = a+bi \longrightarrow \begin{cases} Re[z] = a \\ Im[z] = b \end{cases}$

Complex Numbers have a complex conjugate, denoted with an asterisk:

$$z = a + bi \qquad z^* = a - bi$$

Complex Numbers also have a norm:

$$|z| = \sqrt{|z|^2} \quad \text{where} \quad |z|^2 = z \times z^*$$

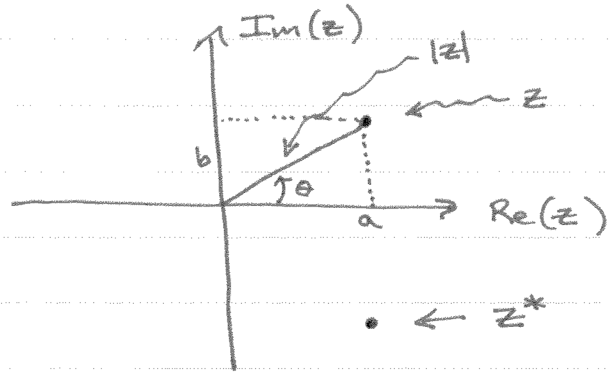
$$= (a+bi)(a-bi)$$

$$= a^2 - bai + abi - b^2 i^2$$

$$|z|^2 = a^2 + b^2$$

So, $|z| = \sqrt{a^2 + b^2}$

Another way to think about complex numbers involves a plane where 1 axis is the real axis and the other is the imaginary axis:



Instead of x- and y- coordinates, we could also use radial coordinates

$$z = \underbrace{|z|\cos\theta}_{= a} + i\,\underbrace{|z|\sin\theta}_{= b}$$

$$z = |z|\, e^{i\theta}$$

$i\theta$ ← phase

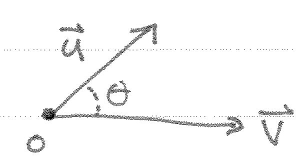
$|z|$ ← magnitude

To do this last step, we have used $\boxed{\text{Euler's relation}}$:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

(what is $e^{-i\pi}$?)

Vectors



$\vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ $\qquad$ $\vec{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$

Two vectors in 3-D space

Each has 3 components describing a direction in space or a displacement from some origin O

The scalar or inner product of 2 vectors:

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$
$$= u_x v_x + u_y v_y + u_z v_z$$

The length of a vector:

$$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{u_x^2 + u_y^2 + u_z^2}$$

looks a lot like the magnitude of a complex number...

Unitvectors have a length of 1:

$$|\vec{u}| = 1$$

these vectors are said to be normalized.

Orthogonal vectors: $\vec{u} \cdot \vec{v} = 0$

$$= |\vec{u}||\vec{v}| \cos\theta$$

implies $\theta = \frac{\pi}{2}$

Orthonormal vectors:

$$\vec{u} \cdot \vec{v} = 0$$
$$\vec{u} \cdot \vec{u} = 1$$
$$\vec{v} \cdot \vec{v} = 1$$

Partial Derivatives. Suppose we have a function of 2 variables, $f(x,y)$

$\frac{\partial f}{\partial x}$ = Hold y constant and take a derivative with respect to x

$\frac{\partial f}{\partial y}$ = Hold x constant & take a derivative wrt y

Here's an example: $f(x,y) = x^2 + 2xy^2 - y^3$

$$\frac{\partial f}{\partial x} = 2x + 2y^2$$
$$\frac{\partial f}{\partial y} = 4xy - 3y^2$$

Operators:

Does something to a function, number, or vector

Examples:

$$2x, \quad \sqrt{\ }, \quad \frac{\partial}{\partial x}, \quad \text{inversion}, \quad \text{rotation}$$

The ordering of operators is important because if we reverse the order, we don't always get the same results!

two operators (with hats)

$$\hat{O} = \frac{\partial}{\partial x} \qquad \hat{P} = x$$

Suppose we have a function $f(x,y) = x^2 + y^2 + 2x^4 y$

$$\hat{O}\hat{P}f = \hat{O}[x^3 + xy^2 + 2x^5 y]$$
$$= 3x^2 + y^2 + 10x^4 y$$

$$\hat{P}\hat{O}f = \hat{P}[2x + 8x^3 y)$$
$$= 2x^2 + 8x^4 y$$

The commutator $[\hat{O}, \hat{P}] = \hat{O}\hat{P} - \hat{P}\hat{O}$

If $[\hat{O},\hat{P}] = 0$ for all functions f, then we can say that operators $\hat{O}$ & $\hat{P}$ commute.

Linear operators have 2 important properties:

1) $\hat{P}(f+g) = \hat{P}f + \hat{P}g$

2) $\hat{P}(a\,f) = a\hat{P}f$

scalar function

$\frac{\partial}{\partial x}$ is a linear operator, $\sqrt{\ }$ is not

Rotation operators

$$\underline{\underline{R}}_{xy}^{\phi} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

rotate by an angle ϕ
in the x, y plane

Rotation operators are matrices that operate on vectors:

$$\vec{u}\,' = \underline{\underline{R}}_{xy}^{\phi}\ \vec{u}$$

original vector

$$= \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$$

$$= \begin{pmatrix} u_x \cos\phi + u_y \sin\phi \\ -u_x \sin\phi + u_y \cos\phi \\ u_z \end{pmatrix}$$

rotated vector!

Eigenvalue equations

$$\hat{P}\ f = P_f\ f$$

the same function as on the left hand side!

operator — function — eigenvalue for function f

example:

$$\hat{P} = \frac{\partial^2}{\partial x^2} \qquad f = \sin ax$$

$$\hat{P} f = -a^2 \sin ax = -a^2 f$$

- $\sin(ax)$ is an eigenfunction of $\frac{\partial^2}{\partial x^2}$ with eigenvalue $-a^2$
- $\cos(ax)$ is also an eigenfunction of $\frac{\partial^2}{\partial x^2}$, also with eigenvalue $-a^2$
- These are degenerate eigenfunctions (they share eigenvalues)

Question: What other functions are also eigenfunctions of this operator? Is it a linear operator?

Question: What if $\hat{p} = \frac{\partial}{\partial x}$

$$\hat{P}\sin(ax) = -a\cos(ax) \longrightarrow \text{not an eigenfunct}$$

But if $f = e^{iax}$

$$\hat{p} f = ia\, e^{iax} \longleftarrow \text{this is!}$$

Central Paradox of QM:

- small enough particles act like <u>waves</u> (e^-)
- wave-like phenomena sometimes act like particles (Photons)

- particle-like behavior: $P_{12} = P_1 + P_2$ ← probabilit[ies] add
- wave-like behavior: $I_{12} = |h_1 + h_2|^2$ ← waves add th[en] <u>square</u>

(particles pick one path <u>or</u> another
waves <u>interfere</u>, so the intensity takes interference into account)

What do waves act like?

- <u>Classical</u> Wave Equation
-

Before we get there, however, some math review

Complex #s
Vectors
Orthonormality
Partial Derivatives
Operators
Linear operators
Eigenvalue equations

A 1-lecture review of last semester

The Classical Wave Equation

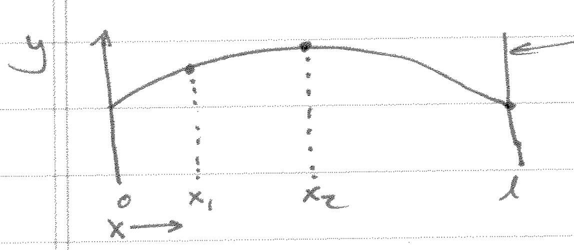
We'll be coming to the Schrödinger Equation soon, but here's a look at the time-dependent Schrödinger equation in 1-D:

$$\boxed{i\hbar \frac{\partial}{\partial t} \psi(x,t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)\right] \psi(x,t)}$$

On top of being an operator equation ($\frac{\partial}{\partial t}$, $\frac{\partial^2}{\partial x^2}$, and $V(x)$ are all operators), it is also a partial differential equation with a complex-valued function, $\psi(x,t)$.

It is also very similar to the classical wave equation, and much of quantum mechanics originates in the fundamental properties of wave motion.

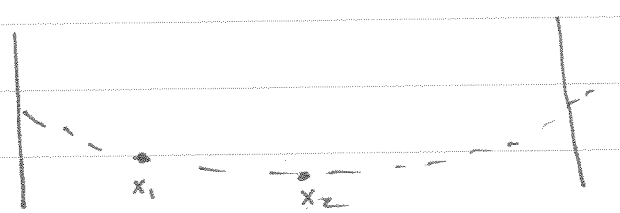
Consider a string stretched tightly between two walls.



At $t=0$

At any particular time, the y-positions of parts of the string at x_1, x_2 are not the same so $y = y(x)$ or y is a function of x.

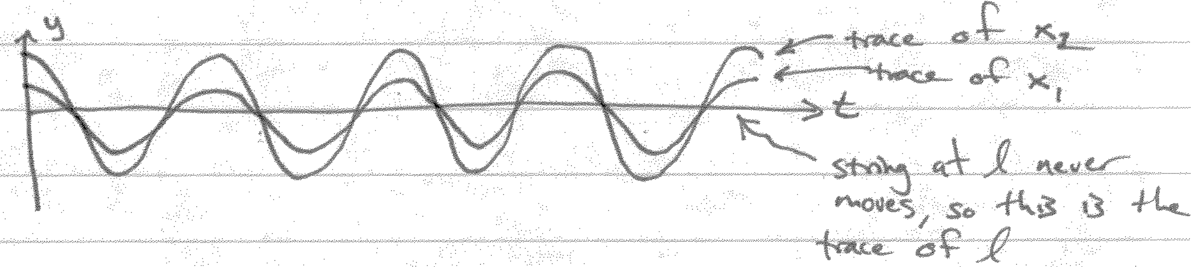
If we let the string go, what happens?



At $t=t'$

What we see now is that the y-positions at x_1 & x_2 are not fixed. They oscillate in time.

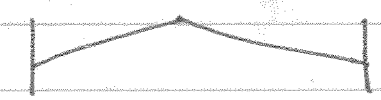
We can plot time traces of these special points:



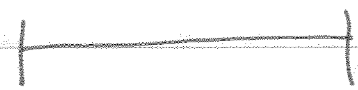
So: $y = y(x,t)$ ← y is a function of both x & t

The Classical Wave Equation (or CWE) shows us how these behave.

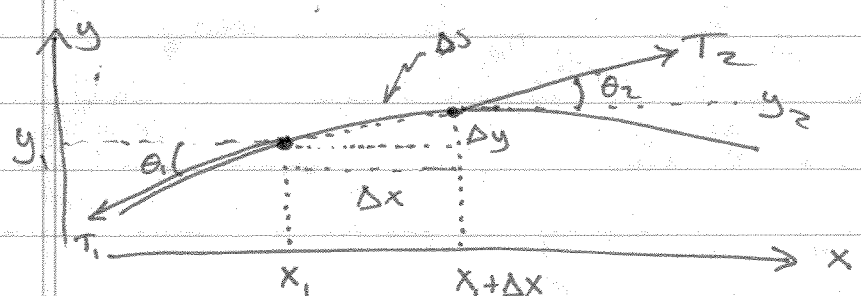
How will this string (non-moving at $t=0$) behave?



What about this one? →



Derivation of the CWE



A section of string in the vicinity of points $(x_1\ y_1)$ and $(x_2\ y_2)$

Assumptions:
1) For the segment of string, the only forces are the 2 tensions T_1 & T_2 (no gravity)
2) String density is the same everywhere.
3) The string only moves vertically ← not required but simpler
4) $\frac{\partial y}{\partial x}$ & $y(x,t)$ are small (string is nearly flat)

We'll use some old school physics, $\boxed{F=ma}$ which has been with us since 1687, to solve this.

First we need the mass of the string segment.

density of string

$$m = \rho \underbrace{\Delta S}$$ length of segment

$$= \rho \sqrt{(\Delta x)^2 + (\Delta y)^2} = \rho \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

Using assumption 4 (the string is nearly flat, $\frac{\Delta y}{\Delta x} \ll 1$, so)

$$\boxed{m \approx \rho \Delta x}$$

Now, on to the forces.

The horizontal components of T_1 & T_2 must be equal or the whole string would move left or right :

$$T_1 \cos\theta_1 = T_2 \cos\theta_2$$

Since the string is nearly flat $\cos\theta_1 \approx \cos\theta_2 \approx 1$

so: * $\boxed{T_1 \cos\theta_1 \approx T_2 \cos\theta_2 \approx T}$ ← total tension on the string.

The interesting forces are the vertical components

$$-T_1 \sin\theta_1 \quad \text{at point } 1$$

$$T_2 \sin\theta_2 \quad \text{at point } 2$$

Total force on segment:

$$F = T_2 \sin\theta_2 - T_1 \sin\theta_1$$

A sneaky trick (multiply by 1 or $\frac{T}{T}$):

$$F = \frac{T}{T} T_2 \sin\theta_2 - \frac{T}{T} T_1 \sin\theta_1$$

Substitute from the * equation above:

$$F \cong \frac{T}{T_2 \cos\theta_2} T_2 \sin\theta_2 - \frac{T}{T_1 \cos\theta_1} T_1 \sin\theta_1$$

$$= T \tan\theta_2 - T \tan\theta_1$$

$$\boxed{F = T \left(\tan\theta_2 - \tan\theta_1 \right)}$$

Now $\underline{\tan\theta_2}$ is the $\underline{\text{slope}}$ of the string at $\underline{x+\Delta x}$, so expressing this in different symbols:

$$F = T \left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x \right]$$

Newton's 2^{nd} law tells us: $F = ma$

$$F = \underbrace{(\rho \Delta x)}_{\text{from before}} \underbrace{\frac{\partial^2 y}{\partial t^2}}_{\text{definition of vertical accelleration of the string}}$$

$$T \left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x \right] = \rho \Delta x \frac{\partial^2 y}{\partial t^2}$$

Let's do a simple re arrangement

$$\frac{\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x \right]}{\Delta x} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2}$$

Now, if we take a limit as the segment gets very small, $\Delta x \to 0$, we recognize the left hand side as a second derivative

$$\lim_{\Delta x \to 0} \frac{\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x \right]}{\Delta x} \longrightarrow \frac{\partial^2 y}{\partial x^2}$$

So:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2}$$

We normally see this written in terms of the velocity of a wave in a string: $v = \sqrt{\frac{T}{\rho}}$

$$\boxed{\frac{\partial^2}{\partial x^2} y(x,t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} y(x,t)} \longleftarrow \text{CWE!}$$

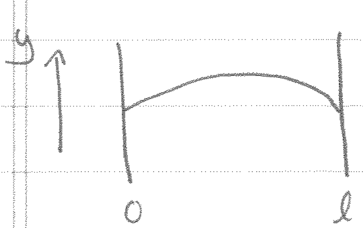
v is the speed a disturbance moves along the string

$\frac{\partial}{\partial x}$ is a partial derivative, so the CWE is called a partial differential equation (PDE)

x & t are independent variables

y is a dependent variable

If the string is stretched between 2 walls:



$$\left.\begin{aligned} y(0,t) &= 0 \\ y(l,t) &= 0 \end{aligned}\right\}$$ These are boundary conditions.

The CWE tells us that there's a relationship between <u>accelleration</u> $\left(\frac{\partial^2 y}{\partial t^2}\right)$ and <u>curvature</u> $\left(\frac{\partial^2 y}{\partial x^2}\right)$

$$\boxed{\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}}$$

This is the "rulebook" for wave motion, much like $F = ma$ is the "rulebook" for particle motion.

If we want to know how a <u>particular</u> string behaves we need to <u>solve</u> this differential equation.

<u>Separating the variables</u>

The first trick we'll use is "separation of variables"

$$y(x,t) = X(x)\,T(t)$$

which makes the assumption that we can factor out the spatial $(X(x))$ behavior from the time-dependent $T(t)$ part:

We do this because the factored functions sail right through partial derivatives. Here's an <u>example</u>

$$T(t) = t + 4$$
$$X(x) = x^2 + 2x + 3$$

Assume: $y(x,t) = X(x)\,T(t) = (x^2 + 2x + 3)(t+4)$

$$= x^2 t + 2xt + 3t + 4x^2 + 8x + 12$$

Now, lets consider what happens under a partial derivative:

$$y(x,t) = x^2t + 2xt + 3t + 4x^2 + 8x + 12$$

$$\frac{\partial y}{\partial x} = 2xt + 2t + 8x + 8$$
$$= x(2t+8) + 1(2t+8)$$
$$= 2x(t+4) + 2(t+4)$$
$$= (2x+2)(t+4)$$
$$= \frac{\partial X(x)}{\partial x} T(t)$$

$$\frac{\partial y}{\partial t} = x^2 + 2x + 3$$
$$= X(x) \cdot 1$$
$$= X(x) \frac{\partial T(t)}{\partial t}$$

So, when we have partial derivatives, we can separate variables to factor out time & spatial dependence into separate equations.

$$y(x,t) = X(x)\,T(t)$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} X(x)T(t)\right] = \frac{\partial}{\partial x}\left[T(t)\frac{\partial X(x)}{\partial x}\right]$$
$$= T(t)\frac{\partial^2 X(x)}{\partial x^2}$$

Likewise:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t} X(x)T(t)\right] = \frac{\partial}{\partial t}\left[X(x)\frac{\partial T(t)}{\partial t}\right]$$
$$= X(x)\frac{\partial^2 T(t)}{\partial t^2}$$

So, applying this to the Classical Wave Equation, we get:

$$T(t)\frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{v^2} X(x)\frac{\partial^2 T(t)}{\partial t^2}$$

It looks like we just made it worse!

But now we can <u>divide</u> both sides by the whole function $y(x,t) = X(k)T(t)$

$$\underbrace{\frac{1}{X(x)} \frac{\partial^2 X(x)}{\partial x^2}}_{\text{contains } \underline{\text{all}} \text{ of the x-dependence}} = \underbrace{\frac{1}{v^2} \frac{1}{T(t)} \frac{\partial^2 T(t)}{\partial t^2}}_{\text{contains } \underline{\text{all}} \text{ of the t-dependence}}$$

The most subtle & hardest to understand part is this: This <u>separated</u> form must be true for all values of x <u>and</u> for all times t. The only way for that to be true is if both sides are <u>constant</u>.

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = -k$$

k is called a separation constant. It is negative by convention.

$$\frac{1}{v^2 T(t)} \frac{d^2 T(t)}{dt^2} = -k$$

We can re write these as:

$$\frac{d^2 X(x)}{dx^2} + k X(x) = 0$$

$$\frac{d^2 T(t)}{dt^2} + k v^2 T(t) = 0$$

Two <u>ordinary</u> 2^{nd} order differential equations that are <u>linked</u> by k.

Now, to solve these, guess

$$X(x) = e^{\alpha x}$$

$$\frac{d}{dx} e^{\alpha x} = \alpha e^{\alpha x} = \alpha X(x)$$

$$\frac{d^2}{dx^2} e^{\alpha x} = \alpha^2 e^{\alpha x} = \alpha^2 X(x)$$

$$\alpha^2 X(x) + k X(x) = 0$$

$$(\alpha^2 + k) X(x) = 0$$

$X(x) = 0$ is called the "trivial" solution

$\alpha^2 + k = 0$ gives us the solution we want

$$\alpha^2 = -k$$

$$\alpha = \pm \sqrt{-k} = \pm \sqrt{-1} \sqrt{k}$$

$$\alpha = \pm i\sqrt{k}$$ ← 2 <u>complex</u> roots

$$X(x) = e^{\pm i\sqrt{k}x}$$ ← 2 <u>solutions</u>

$$X(x) = \frac{e^{+i\sqrt{k}x} + e^{-i\sqrt{k}x}}{2} = \cos(\sqrt{k}x)$$ is <u>also</u> a solution

$$X(x) = \frac{e^{i\sqrt{k}x} - e^{-i\sqrt{k}x}}{2i} = \sin(\sqrt{k}x)$$ is <u>also</u> a solution

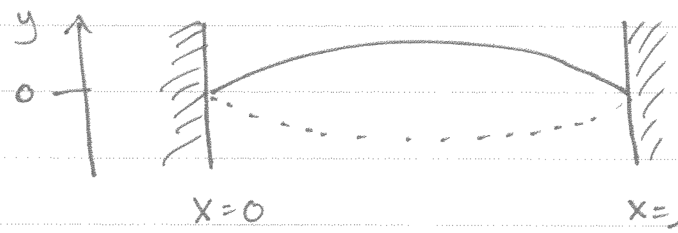
<u>Generally:</u>

$$X(x) = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x)$$

These are called the <u>general</u> solutions to the spatial part of the wave equation. A & B are still unknown numbers at this point.

To find them, we need to use the boundary conditions:

Boundary conditions:



$$\left.\begin{aligned} y(0,t) &= 0 \\ y(l,t) &= 0 \end{aligned}\right\} \text{ fixed boundaries}$$

This boundary condition tells us that we can't have any $\cos(x)$ contribution because $\cos(0) = 1$, so we know that $\underline{A = 0}$

$$\therefore \quad X(x) = B \sin\left(\sqrt{k}\, x\right)$$

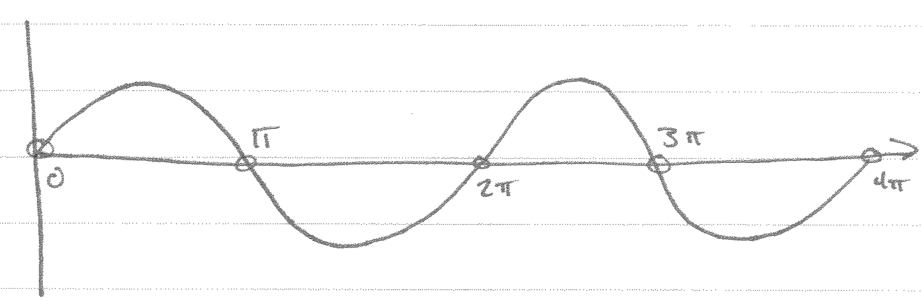
2 unknowns

What about the <u>other</u> boundary?

$$X(l) = 0 = B \sin\left(\sqrt{k}\, l\right)$$

$$\therefore \quad \sin\left(\sqrt{k}\, l\right) = 0$$

The sin function <u>only</u> has zeroes at specific values: $x = n\pi \longrightarrow \sin(x) = 0$ $\quad$ n is the integers



$\therefore$ to satisfy the other boundary condition, $x = n\pi$

$$\sqrt{k}\, l = n\pi$$

$$\sqrt{k} = \frac{n\pi}{l} \longrightarrow k = \frac{n^2\pi^2}{l^2}$$

So now we only have 1 unknown:

$$\boxed{X(x) = B \sin\left(\frac{n\pi x}{l}\right)}$$

← This function satisfies <u>both</u> boundary conditions for any value of n that is an <u>integer</u>

Now, on to time dependence:

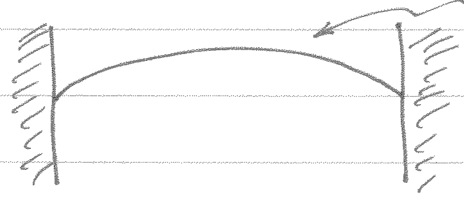
$$\frac{d^2T(t)}{dt^2} + k v^2 T(t) = 0$$

← remember that $\sqrt{k} = \frac{n\pi}{l}$

This is a very similar equation to the spatial $(X(x))$ equation, so a similar solution will probably work here also.

What are the <u>boundary conditions</u> on $T(t)$?

If we pull a string that we release when $t=0$



Stationary at $t=0$

$$\frac{dy(x,t)}{dt} = 0 \quad \text{at } t=0 \text{ for } \underline{\text{all}}\ x$$

$$\therefore \frac{dT(t)}{dt} = 0 \quad \text{at } t=0$$

$$\frac{d^2T(t)}{dt^2} + \underbrace{\frac{n^2\pi^2}{l^2} v^2}_{= \omega_n^2} T(t) = 0$$

← each value of n gives a different constant

$T(t) = e^{\alpha t}$

$$(\alpha^2 + \omega_n^2) T(t) = 0 \longrightarrow \alpha^2 = -\omega_n^2$$

$$\alpha = \pm i \omega_n$$

Last time:

CWE : $$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

The "rules" containing all the physics (F=ma

Separation of variables: $y(x,t) = X(x)T(t)$

is what happens to an actual & particular string

$$\frac{d^2X(x)}{dx^2} + kX(x) = 0$$
$$\frac{d^2T(t)}{dt^2} + kv^2T(t) = 0$$

separate 2nd order differential eqs.

Solution for $X(x)$:

$$X(x) = A\cos(\sqrt{k}\,x) + B\sin(\sqrt{k}\,x)$$

Apply boundaries : $X(0) = 0 \longrightarrow A = 0$

$X(l) = 0 \longrightarrow \sqrt{k}\,l = n\pi$

$\sqrt{k} = \frac{n\pi}{l}$

$$\therefore \; X(x) = B\sin\left(\frac{n\pi x}{l}\right)$$

Now, on to time dependences (remember that $\sqrt{k} = \frac{n\pi}{l}$)

$$\frac{d^2T(t)}{dt^2} + kv^2T(t) = 0$$

$$\frac{d^2T(t)}{dt^2} + \underbrace{\frac{n^2\pi^2v^2}{l^2}}_{\omega_n^2} T(t) = 0$$

$$\omega_n = \frac{n\pi v}{l}$$

to solve this we assume $T(t) = e^{\alpha t}$

$\hookrightarrow T' = \alpha e^{\alpha t} \rightarrow T'' = \alpha^2 e^{\alpha t}$

$$\alpha^2 T(t) + \omega_n^2 T(t) = 0$$

$$(\alpha^2 + \omega_n^2) T(t) = 0$$

$$\alpha^2 + \omega_n^2 = 0$$

$$\alpha^2 = -\omega_n^2$$

$$\alpha = \pm\sqrt{-1}\sqrt{\omega_n^2}$$

$$\alpha = \pm i\omega_n$$

$\therefore$

$T(t) = e^{i\omega_n t}$ or $e^{-i\omega_n t}$ or any combination of these

$\therefore$

$$T(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$$

Let's verify this quickly:

$$T'(t) = -\omega_n C_1 \sin(\omega_n t) + \omega_n C_2 \cos(\omega_n t)$$

$$T''(t) = -\omega_n^2 C_1 \cos(\omega_n t) - \omega_n^2 C_2 \sin(\omega_n t)$$

$$= -\omega_n^2 T(t)$$

$$\therefore \frac{d^2T(t)}{dt^2} + \omega_n^2 T(t) = -\omega_n^2 T(t) + \omega_n^2 T(t) = 0$$

So, we've just verified that

$$T(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$$

works in the time differential equation when $\omega_n = \frac{n\pi v}{l}$

We can put the whole thing together:

$$y_n(x,t) = \underbrace{B \sin\left(\frac{n\pi x}{l}\right)}_{X(x)} \underbrace{\left[C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)\right]}_{T(t)}$$

Combining coefficients, we get:

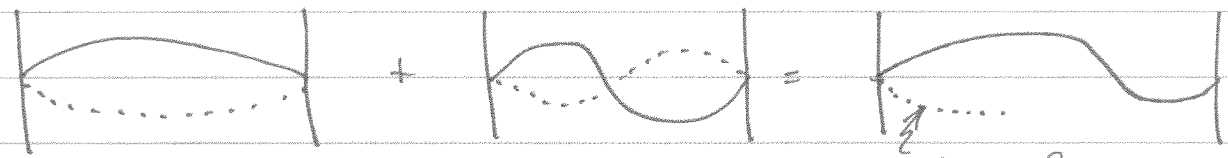
$$y_n(x,t) = \left[C_{1,n} \cos(\omega_n t) + C_{2,n} \sin(\omega_n t)\right] \sin\left(\frac{n\pi x}{l}\right)$$

where $n = 1, 2, 3, \ldots$
and $\omega_n = \frac{n\pi v}{l}$

This gives us an <u>infinite</u> series of possible solutions. The $n=1$ solution is called the <u>fundament</u>
Higher n solutions are called "harmonics"

Math	ω_n	music	note
$n=1$	$v\pi/l$	[fundamental or 1st harmonic]	C
$n=2$	$2v\pi/l$	[1st overtone or 2nd harmonic]	C (octave)
$n=3$	$3\pi v/l$	2nd overtone	G (major 5th)
$n=4$	$4\pi v/l$		C (2 octaves)
$n=5$	$5\pi v/l$		E (major 3rd)

Real sounds (and wavefunctions) are <u>combinations</u> or <u>hybrids</u> of these harmonics



$\omega = \frac{v\pi}{\ell}$ $\qquad$ $\omega = \frac{2v\pi}{\ell}$ $\qquad$ still satisfies boundary conditions

$$y(x,t) = \left[F_1 \cos(\omega_1 t) + G_1 \sin(\omega_1 t)\right] \sin\frac{\pi x}{\ell} + \left[F_2 \cos \omega_2 t + G_2 \sin \omega_2 t\right] \sin\frac{2\pi x}{\ell}$$

Most generally:

$$y(x,t) = \sum_{n=1}^{\infty} \left[F_n \cos \omega_n t + G_n \sin \omega_n t\right] \sin\frac{n\pi x}{\ell}$$

We can always rewrite sums of sin + cos with the same argument in a simpler form. This is a problem on problem set 2.

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} A_n \cos(\omega_n t + \phi_n) \sin\left(\frac{n\pi x}{\ell}\right)}$$

<u>amplitude</u> of mode n

<u>frequency</u> of mode n $\omega_n = \frac{vn\pi}{\ell}$

<u>phase</u> of mode n (set by initial conditions)

length of string

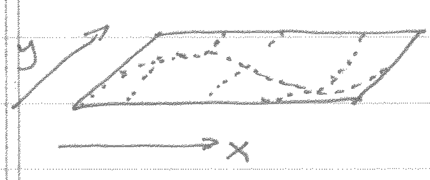
And that's the <u>solution</u> to the classical wave equation in 1-D. We've left some things out

1) Friction (Damps the string at long times)
2) Bowing (forcing the string into an oscillation)
3) "Fretting" touching the string at a nodal point

In 2 dimensions

Membranes or drumheads vibrating between clamped edges.

$u(x, y, t)$ = height of membrane at position (x, y) and time t



$$\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 u(x,y,t)}{\partial t^2}$$

This is the classical wave equation in 2D.

How do we solve it?

First, separation of variables:

$$u(x,y,t) = X(x)\,Y(y)\,T(t)$$

why?

$$\frac{\partial^2 u}{\partial x^2} = Y(y)\,T(t)\,\frac{\partial^2 X(x)}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = X(x)\,T(t)\,\frac{\partial^2 Y(y)}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial t^2} = X(x)\,Y(y)\,\frac{\partial^2 T(t)}{\partial t^2}$$

partial derivatives only act on one of the functions

In the CWE, these become:

$$Y(y)\,T(t)\,\frac{\partial^2 X(x)}{\partial x^2} + X(x)\,T(t)\,\frac{\partial^2 Y(y)}{\partial y^2} = \frac{1}{v^2} X(x)\,Y(y)\,\frac{\partial^2 T(t)}{\partial t^2}$$

divide by $u = X(x)\,Y(y)\,T(t)$

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = \frac{1}{v^2 T(t)}\frac{\partial^2 T(t)}{\partial t^2}$$

$$\underbrace{\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}}_{\text{all } x} + \underbrace{\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2}}_{\text{all } y} = \underbrace{\frac{1}{v^2 T(t)}\frac{\partial^2 T(t)}{\partial t^2}}_{\text{all } t}$$

$$-p^2 \quad + \quad -q^2 \quad = \quad -s^2$$

three constants

Leads to 3 diff eqs:

$$\frac{d^2 X(x)}{dx^2} + p^2 X(x) = 0$$

$$\frac{d^2 Y(y)}{dy^2} + q^2 Y(y) = 0$$

$$\frac{d^2 T(t)}{dt^2} + v^2 s^2 T(t) = 0$$

All are nearly identical to the 1D 2nd order Differential equation

General solutions

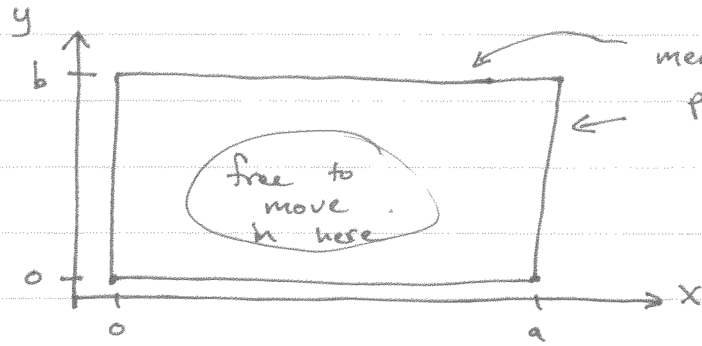
$$X(x) = A\cos(px) + B\sin(px)$$

$$Y(y) = C\cos(qy) + D\sin(qy)$$

$$T(t) = E\cos(vst) + F\sin(vst)$$

Boundary conditions in 2D:

for a rectangular membrane



membrane is pinned at the edges

$X(0) = 0 \qquad X(a) = 0$

$Y(0) = 0 \qquad Y(b) = 0$

$$\therefore \quad X(x) = B \sin\left(\frac{n\pi x}{a}\right) \qquad p = \frac{n\pi}{a}$$

$$Y(y) = D \sin\left(\frac{m\pi y}{b}\right) \qquad q = \frac{m\pi}{b}$$

n & m are both integers $\quad n = 1, 2, 3, \ldots \quad m = 1, 2, 3, \ldots$

We also know

$$p^2 + q^2 = s^2$$

$$\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2} = s^2 = \frac{\omega_{nm}^2}{V^2}$$

we're going to define ω_{nm} in terms of n & m:

$$\omega_{nm}^2 = V^2\pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)$$

$$\omega_{nm} = V\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \underbrace{A_{nm}}_{\text{Amplitude}} (\cos \underbrace{\omega_{nm}}_{\text{frequency}} t + \underbrace{\phi_{nm}}_{\text{phase}}) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

frequency $= \omega_{nm} = V\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$

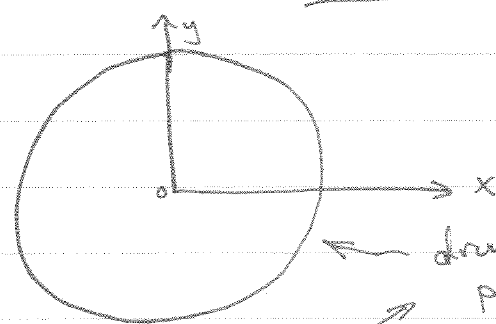
For a square membrane

$$\omega_{nm} = \frac{v\pi}{a}\sqrt{n^2+m^2}$$

$\therefore$ $\omega_{21} = \omega_{12}$ = doubly degenerate

(frequencies of 2 modes are the same)

What happens for non-rectangular boundaries



$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

This is still the rule book

drumhead pinned at $x^2+y^2 = R^2$

boundary conditions are not as amenable to simple solution.

In this case we have to replace

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \longrightarrow \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$\boxed{\frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2} = \frac{1}{v^2}\frac{\partial^2 u(r,\theta,t)}{\partial t^2}}$$

The same classical wave equation, but recast in cylindrical coordinates.