RG theory

What's wrong with MFT? It neglects correlations between spins:

$$J \sum_{n,n'} \sigma_n \sigma_{n'} \longrightarrow NJ2ds$$

But those correlations make the exact solution hard. Is there a way to make the effective correlations very weak?

$$Q(K,N) = \sum_{\sigma_1 \cdots \sigma_N = \pm 1} e^{K(\sigma_1\sigma_2 + \sigma_2\sigma_3 \cdots + \sigma_{N-1}\sigma_N + \sigma_N\sigma_1)}$$

$\Big\{$ 1-D, PBC, with $K = \frac{J}{k_B T}$

RG theory takes the full system and creates a mapping to a new system with a smaller K

Here's how:

$$Q(K,N) = \sum_{\{\}} \underbrace{e^{K(\sigma_1\sigma_2 + \sigma_2\sigma_3)}}_{\text{all } \sigma_2 \text{ here}} \underbrace{e^{K(\sigma_3\sigma_4 + \sigma_4\sigma_5)}}_{\text{all } \sigma_4 \text{ here}} \underbrace{e^{K(\sigma_5\sigma_6 + \sigma_6\sigma_7)}}_{\text{all } \sigma_6 \text{ here}} \cdots$$

Suppose we sum over these even spins:

$$\sum_{\sigma_2 = \pm 1} e^{K(\sigma_1\sigma_2 + \sigma_2\sigma_3)} = e^{K(\sigma_1 + \sigma_3)} + e^{-K(\sigma_1 + \sigma_3)}$$

Likewise for σ_4: $\displaystyle\sum_{\sigma_4 = \pm 1} e^{K(\sigma_3\sigma_4 + \sigma_4\sigma_5)} = e^{K(\sigma_3 + \sigma_5)} + e^{-K(\sigma_3 + \sigma_5)}$

If we put this back together:

$$Q(K,N) = \sum_{\sigma_1 \sigma_3 \sigma_5 = \pm 1} \left[e^{K(\sigma_1+\sigma_3)} + e^{-K(\sigma_1+\sigma_3)} \right] \left[e^{K(\sigma_3+\sigma_5)} + e^{-K(\sigma_3+\sigma_5)} \right] \cdots$$

$$\underset{1}{\bullet}\ \underset{2}{\bullet}\ \underset{3}{\bullet}\ \underset{4}{\bullet}\ \underset{5}{\bullet}\ \underset{6}{\bullet}\ \underset{7}{\bullet} \longrightarrow \underset{1}{\bullet}\quad \underset{3}{\bullet}\quad \underset{5}{\bullet}\quad \underset{7}{\bullet}$$

The idea is that this looks like a partition function for a system with <u>only</u> the odd spins (and a different Hamiltonian)

We'd like to make it look <u>just like</u> the original system, but to do that we'll need

$$e^{Ks_1 s_2} e^{Ks_2 s_3} \longrightarrow e^{K(s+s')} + e^{-K(s+s')} = \underbrace{f(K)}_{\text{a transformation function}} e^{K' s s'}$$

$$Q(K,N) = \sum_{\sigma_1 \sigma_3 \sigma_5 = \pm 1} f(K) e^{K'\sigma_1\sigma_3} f(K) e^{K'\sigma_3\sigma_5} f(K) e^{K'\sigma_5\sigma_7} \cdots$$

$$\underbrace{Q(K,N) = f(K)^{N/2}\, Q\left(K', \frac{N}{2}\right)}_{\text{a Kadanoff transformation}} \longleftarrow \text{a self-similar partition function}$$

So to use this property, (self similarity & recursion)
we need:

$$f(K) \text{ and } K'$$

$$e^{K(s+s')} + e^{-K(s+s')} = f(K)\, e^{K'ss'}$$

Suppose $s = s' = +1$:

$$e^{2K} + e^{-2K} = f(K)\, e^{K'}$$

Suppose $s = +1$, $s' = -1$

$$e^{0} + e^{0} = f(K)\, e^{-K'}$$

So we have 2 equations & 2 unknowns:

$$2 = f(K)\, e^{-K'}$$

$$2\cosh 2K = f(K)\, e^{K'}$$

$$\ln 2\cosh 2K = \ln f(K) + K'$$

$$\ln 2 = \ln f(K) - K'$$

$$2 \ln f(K) = \ln 2\cosh 2K + \ln 2$$

$$\ln f(K)^2 = \ln 4\cosh 2K$$

$$f(K)^2 = 4\cosh 2K$$

$$\boxed{f(K) = 2\cosh^{1/2} 2K}$$

$$\ln 2\cosh 2K - \ln 2 = 2K'$$

$$\ln \cosh 2K = 2K'$$

$$\boxed{K' = \frac{1}{2} \ln \cosh 2K}$$

OK, so we've got:

$$Q(K,N) = f(K)^{N/2}\, Q(K', N/2)$$

$$f(K) = 2\cosh^{1/2} 2K$$
$$K' = \tfrac{1}{2}\ln\cosh 2K$$

The idea is to keep repeating this mapping from the N spin system $\rightarrow N/2, \rightarrow N/4 \rightarrow N/8 \rightarrow N/16$ until we get convergence on K.

Let's make the recursion more explicit:

$$g(K) = \frac{1}{N}\ln Q(K,N) \quad \leftarrow \text{related to free energy}$$

$$g(K) = \frac{1}{N}\left(\frac{N}{2}\ln f(K) + \ln Q(K', \tfrac{N}{2})\right)$$

$$= \frac{1}{2}\ln f(K) + \frac{1}{N}\ln Q(K', \tfrac{N}{2})$$

$$2g(K) = \ln f(K) + \underbrace{\frac{2}{N}\ln Q(K', \tfrac{N}{2})}_{\text{definition of } g(K')}$$

$$g(K') = 2g(K) - \ln\left[2\cosh^{1/2}(2K)\right]$$

with

$$K' = \tfrac{1}{2}\ln\cosh 2K$$

This is a recursive way of obtaining the Free energies!

$$A = -Nk_BT\, g(K)$$

Also, these 2 equations can be <u>reversed</u>:

$$K = \frac{1}{2}\cosh^{-1}\left(e^{2K'}\right)$$
$$g(K) = \frac{1}{2}g(K') + \frac{1}{2}\ln 2 + \frac{K'}{2}$$

So starting from a trial coupling, we can iterate to smaller <u>or</u> larger couplings and converge on a value for K:

First set of equations: $K \approx 0 \longleftarrow K \approx \infty$

Second set: $K \approx 0 \longrightarrow K \approx \infty$

These are called RG flow maps.

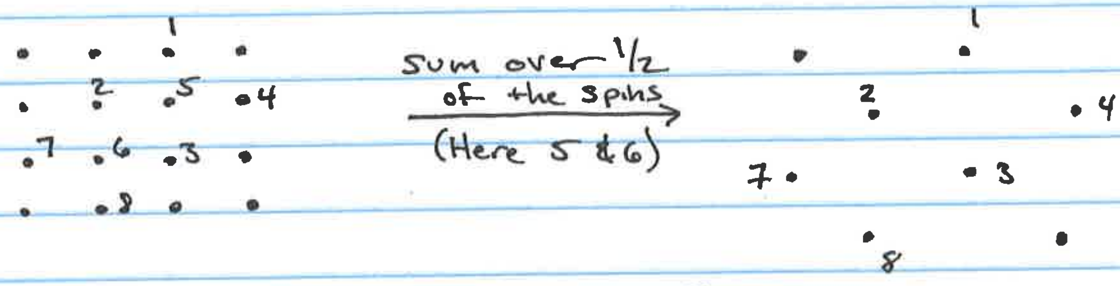
For systems with critical points, the flow maps will have non-trivial fixed points (attractors or repellors):

$$K \approx 0 \longrightarrow K_c \longleftarrow K \approx \infty$$

<u>Or</u>:

$$K \approx 0 \longleftarrow K_c \longrightarrow K \approx \infty$$

RG for 2-D:



This new lattice is still a 2D Ising lattice <u>rotated</u> by 45°

$$Q = \sum_{\{\}} \underbrace{e^{K\sigma_5(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}}_{\text{all the } \sigma_5} \underbrace{e^{K\sigma_6(\sigma_2+\sigma_3+\sigma_7+\sigma_8)}}_{\text{all the } \sigma_6}$$

Summing over spins σ_5 & σ_6 we have

$$Q = \sum_{\{\}} \left[e^{K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} + e^{-K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}\right]\left[e^{K(\sigma_2+\sigma_3+\sigma_7+\sigma_8)} + e^{-K(\sigma_2+\sigma_3+\sigma_7+\sigma_8)}\right]$$

So our transformation will need:

$$e^{K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} + e^{-K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} = f(K)\, e^{K'\underbrace{(\sigma_1\sigma_2+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_3\sigma_4)}_{\text{all the NN combos in the new lattice}}}$$

Now we have <u>4</u> possible combinations:

all ⊕, 3⊕, 1⊖, 2⊕ & 2⊖, 1⊕ & 3⊖,

A simple mapping (like we had in 1D) doesn't exist, but there's a more complicated mapping!

$$e^{K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} + e^{-K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} = f(K)e^{K_1(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)} \times e^{K_2(\sigma_1\sigma_3+\sigma_2\sigma_4)} \times e^{K_3\sigma_1\sigma_2\sigma_3\sigma_4}$$

Now, we have 4 cases & 4 unknowns

$$e^{4K} + e^{-4K} = f(K)e^{2K_1+2K_2+K_3}$$
$$e^{2K} + e^{-2K} = f(K)\,e^{-K_3}$$
$$2 = f(K)\,e^{-2K_2+K_3}$$
$$2 = f(K)\,e^{-2K_1+2K_2+K_3}$$

And this Kadanoff transformation has solutions:

$$K_1 = \frac{1}{4}\ln\cosh(4K)$$
$$K_2 = \frac{1}{8}\ln\cosh(4K)$$
$$K_3 = \frac{1}{8}\ln\cosh(4K) - \frac{1}{2}\ln[\cosh 2K]$$

$$f(K) = 2[2\cosh(2K)]^{1/2}[\cosh 4K]^{1/8}$$

The final approach illustrates convergence:

$$g(K) = \frac{1}{2}\ln f(K) + \frac{1}{2}g(K')$$
$$g(K') = 2g(K) - \ln\left[2\cosh(2K)^{1/2}\cosh(4K)^{1/8}\right]$$

$$K' = \frac{3}{8}\ln[\cosh(4K)]$$

$$K=0 \longleftarrow K_c \longrightarrow K=\infty$$

$$\parallel$$

$$K_c = 0.50698$$

What does this mean?

Remember $K = \frac{J}{k_B T} \Rightarrow K_c = \frac{J}{k_B T_c}$

$$T_c = 1.972 \frac{J}{k_B}$$

(Onsager's result: $T_c = 2.269 \frac{J}{k_B}$)

(MFT $T_c = 4 \frac{J}{k_B}$)