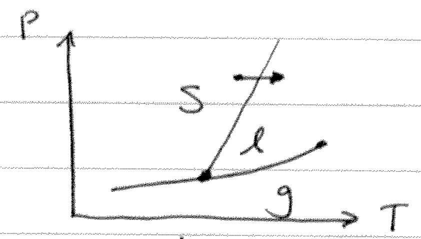
Phase Transitions



$$G(P, T, N_A, N_B \cdots)$$

$$dG = \left(\frac{\partial G}{\partial P}\right) dP + \left(\frac{\partial G}{\partial T}\right) dT + \left(\frac{\partial G}{\partial N_A}\right) dN_A + \left(\frac{\partial G}{\partial N_B}\right) dN_B$$

$$dG = V\,dP - S\,dT + \underbrace{\mu_A\,dN_A + \mu_B\,dN_B}_{+ \sum_i \nu_i \mu_i \, d\lambda}$$

A phase transition happens when two phases are in equilibrium

$$A_{(s)} \rightleftharpoons A_{(l)}$$

We know the condition for equilibrium is that:

$$\sum_i \nu_i \mu_i = 0$$

chemical potential in phase α

or:

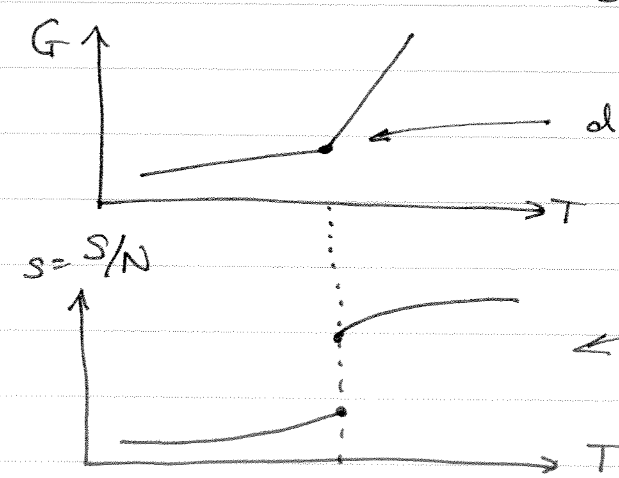
$$\mu_i^{(\alpha)}(T,P) = \mu_i^{(\beta)}(T,P)$$

chemical potential in phase β

$$\mu_A^{(s)}(T,P) = \mu_A^{(l)}(T,P)$$

We also know that $dG = 0$ at a phase boundary (e.g. G is continuous)

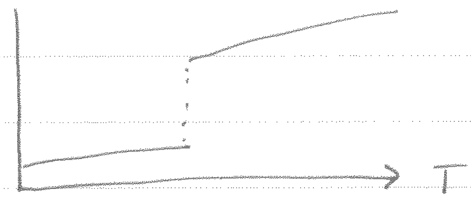
Suppose we come from solid to liquid:



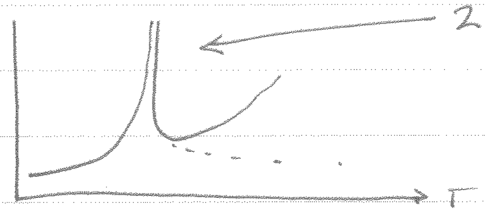
$dG = 0$, but has a different <u>slope</u> on either side of phase boundary

The "derivative" quantities are discontinuous

$v = \frac{V}{N}$



$C_p = \left(\frac{\partial S}{\partial T}\right)_p$



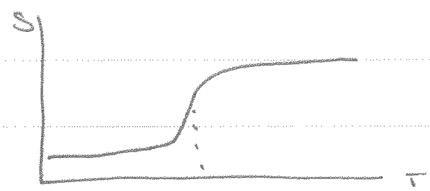
2^{nd} derivative quantity <u>diverges</u> at phase boundary

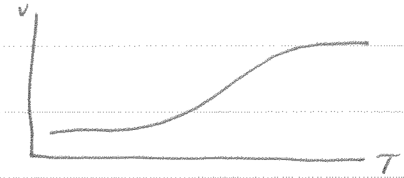
$C_p \approx |T - T_c|^{-\alpha} + \text{const}$

α = critical scaling exponent

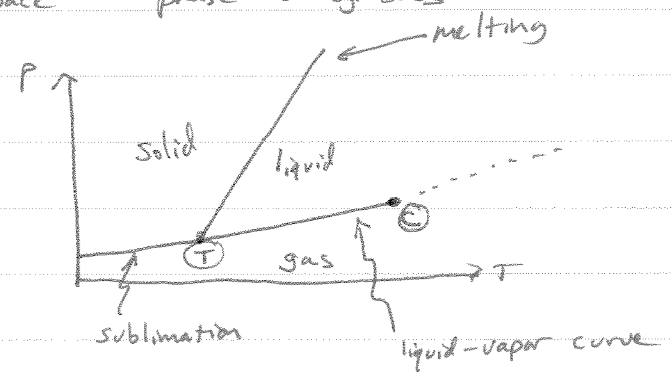
2^{nd} order phase transitions: 1^{st} order derivs are continuous

2^{nd} order derivs are discontinuous



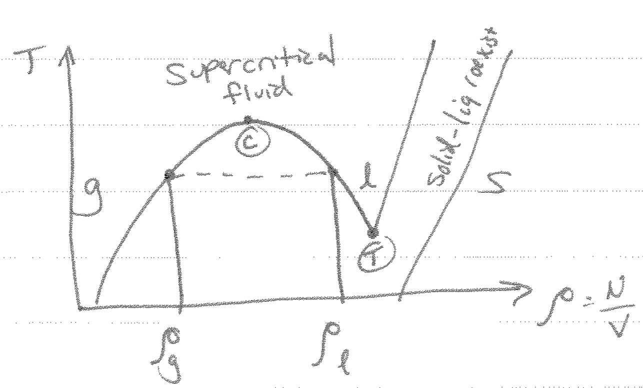


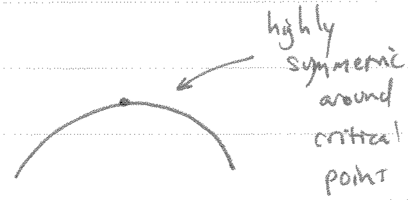
Field-Space phase diagrams



—— = 1^{st} order

- - - - - = 2^{nd} order



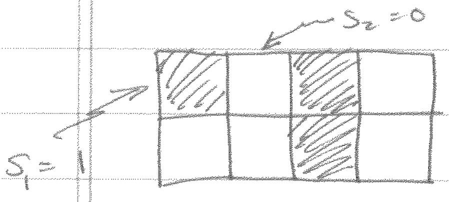


$\Delta\rho = \rho_l - \rho_g = \left(\frac{T - T_c}{T_c}\right)^{\beta}$

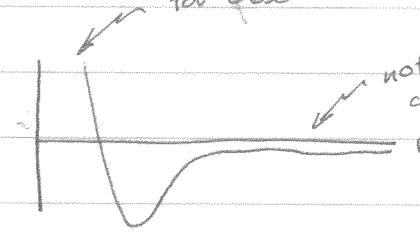
$\beta = 0.326$

(3)

Can we model phase behavior:

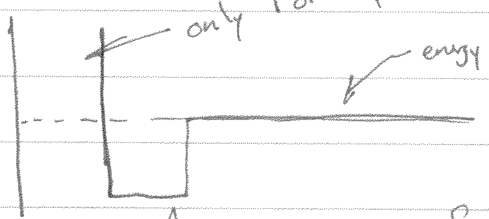


Energy:

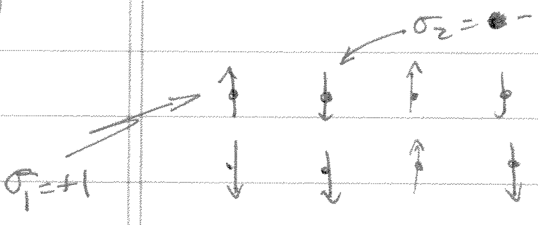


Lattice Gas

(squares occupied or not



The lattice gas is an important model for l-g, l-s phase transitions. It maps exactly onto a model for magnetic phase transitions:



These are exactly the same model after a simple transformation.

$$\mathcal{H} = \underbrace{-\sum_n h_n \sigma_n}_{\text{interactions of spins with external field}} \underbrace{-\frac{1}{2}\sum_n \sum_{n'} J_{n,n'}\, \sigma_n \sigma_{n'}}_{\text{interactions of spins with each other}} \underbrace{-\frac{1}{6}\sum_n \sum_{n'} \sum_{n''} L_{n n' n''}\, \sigma_n \sigma_{n'} \sigma_{n''}}_{\text{3-body interaction}}$$

$$\sigma_n = \pm 1$$

The Ising model is a simplification. We stop at pairwise interactions and only include contributions from nearest neighbors:

$$J_{n,n'} = \begin{cases} 0 & \text{if } n = n' \\ J & \text{if } n \text{ and } n' \text{ are nearest neighbors} \\ 0 & \text{if } n \text{ and } n' \text{ are not} \end{cases}$$

$$H_n = \xi H$$

(4)

$$\mathcal{H}_{Ising} = -H \sum_n \sigma_n - \frac{J}{2} \sum_n \sum_{n'}^{NN} \sigma_n \sigma_{n'}$$

Ising model is <u>discrete</u> $\sigma_n = \pm 1$
and short-ranged.

Basic properties:

What is configuration at $T=0$? (and $H=0$)

With $J>0$:

$\mathcal{H} = \frac{-J}{2} \sum_n \sum_{n'} \sigma_n \sigma_{n'}$

Ferromagnetic:

$\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow$
$\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow$ } all spins point the same direction

$\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow$
$\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow$ } also a ferromagnetic low energy state

These two solutions are <u>symmetric</u>

With $J<0$:

$\uparrow \downarrow \uparrow \downarrow \uparrow$
$\downarrow \uparrow \downarrow \uparrow \downarrow$
$\uparrow \downarrow \uparrow \downarrow \uparrow$ } anti ferromagnetic all spins surrounded by spins going the other direction

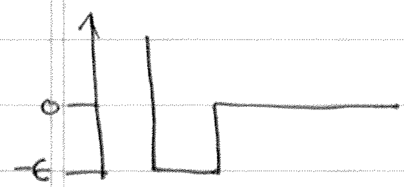
Is this state symmetric?

What Happens if the field is on
$H \neq 0$

We break the symmetry:

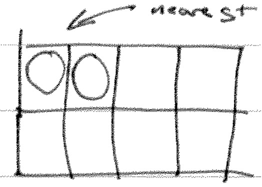
$\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow$ $\quad$ $\downarrow \downarrow \downarrow \downarrow \downarrow$
$\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow$ $\quad$ $\downarrow \downarrow \downarrow \downarrow \downarrow$

Lattice Gas



$$\mathcal{H}_{LG} = \frac{-\epsilon}{2} \sum_{n,n'}^{NN} S_n S_{n'} \quad \leftarrow S_n = 0,1$$

nearest neighbors



$$\mathcal{H}_{ising} = -\frac{J}{2} \sum_{n,n'}^{NN} \sigma_n \sigma_{n'} - H \sum_n \sigma_n \quad \leftarrow \sigma_n = \pm 1$$

$$Q_{ising} = \sum_{\{\sigma_n = \pm 1\}} e^{-\beta \mathcal{H}_{ising}} = \sum_{\{\sigma_n = \pm 1\}} e^{\beta\left(\frac{J}{2} \sum_{nn'} \sigma_n \sigma_{n'} + H \sum_n \sigma_n\right)}$$

energy of that lattice state

sum over all possible states of the lattice

Consider the Grand canonical P.F. for the Lattice Gas

$$\Xi_{LG} = \sum_{\{S_n = 0,1\}} e^{-\beta \mathcal{H}_{LG}} e^{\beta \mu N} = \sum_{\{S_N = 0,1\}} e^{\beta\left(\frac{\epsilon}{2} \sum_{nn'} S_n S_{n'} + \mu \sum_n S_n\right)}$$

$$\therefore \quad Q_{ising} \equiv \Xi_{\text{Lattice Gas}}$$

with
$$J = 2\epsilon - 1$$
$$H = 2\mu - 1$$
$$\sigma_n = 2S_n - 1$$

mapping from canonical ising to grand canonical Lattice Gas

Solve one of these problems and you've solve the other!

What do we want to know?

$$C_V \sim |T - T_c|^{-\alpha} + c$$

$$\Delta\rho = \rho_l - \rho_g \sim \frac{(T - T_c)^{\beta}}{T_c}$$

To get C_V we need $A(T)$

To get $A(T)$ we need Q_{ising}

$$\langle \rho \rangle = \frac{\langle N \rangle}{V} = \frac{1}{V} \frac{\partial \ln \Xi_L}{\partial(\beta\mu)}$$

$$= \frac{1}{V} \frac{1}{\Xi} \sum_{\{s_n = 0,1\}} \left(\sum_n s_n\right) e^{\cdots}$$

What's the equivalent to ρ for the ising model?

$$\langle m \rangle = \frac{1}{Q} \sum_{\text{states}} \left(\sum_n \sigma_n\right) e^{-\beta \cdots}$$

$$= \frac{\partial \ln Q_{ising}}{\partial(\beta H)}$$

← net or bulk magnetization of the ising lattice!

$\langle m \rangle$ = magnetization

for ferromagnetic phases

↑↑↑↑↑
↑↑↑↑↑ $\langle m \rangle = +1$

↓↓↓↓↓↓
↓↓↓↓↓↓ $\langle m \rangle = -1$

anti-ferromagnetic = ↑↓↑↓↑
↓↑↓↑↓ $\langle m \rangle = 0$

random $\langle m \rangle = 0$

A quick note about Frustration: Assume $J < 0$

favors antialignment

↑

↑ ↓

A triangular lattice is frustrated when $J < 0$

Types of Frustration:

<u>Complete</u>: situations like triangular lattice where it is <u>impossible</u> to satisfy microscopic preferences

<u>Partial</u>: involves higher order couplings:

next nearest neighbors

$$\mathcal{H} = -\frac{J_1}{2}\sum_{n,n'}^{NN} \sigma_n \sigma_{n'} - \frac{J_2}{2}\sum_{n,n'}^{NNN} \sigma_n \sigma_{n'}$$

$$\text{If} \quad \underbrace{J_1 > 0}_{\text{prefers NN aligned}} \quad \text{and} \quad \underbrace{J_2 < 0}_{\text{prefers NNN anti-aligned}}$$

these are incommensurate desires

Depending on relative strengths of J_1 & J_2 one will always "win"

<u>Irregular</u> frustration: $\mathcal{H} = -\frac{1}{2}\sum_{n,n'} J_{n,n'} \sigma_n \sigma_{n'}$

Pick values of $J_{n,n'}$ <u>randomly</u> on $[-1, 1]$

<u>Locally</u> frustrated structures depend on random variables

This shows up in spin glasses & neural nets

Ising model Partition Functions

$$\mathcal{H} = -\frac{J}{2} \sum_n \sum_{n'}{}' \sigma_n \sigma_{n'}$$ ← nearest neighbors

In 1-D: (no field, no periodic boundaries)

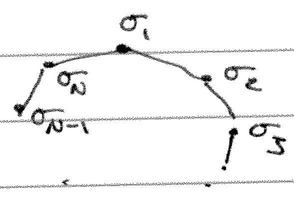
$$\mathcal{H} = -\frac{J}{2}\left(\underbrace{\sigma_1\sigma_2 + (\sigma_2\sigma_1}_{} + \underbrace{\sigma_2\sigma_3) + (\sigma_3\sigma_2}_{} + \underbrace{\sigma_3\sigma_4)}_{} + \dots\right.$$

We can recombine these terms together

$$\mathcal{H} = -J \sum_{n=1}^{N-1} \sigma_n \sigma_{n+1}$$ ← written so each spin only couples to next one in line...

With periodic boundaries:

$$\mathcal{H} = -J \sum_{n=1}^{N} \sigma_n \sigma_{n+1}$$ ← one more term



← edge effects are gone because $\sigma_{N+1} = \sigma_1$

We can define a bond variable $b_i = \sigma_i \sigma_{i+1}$

It has values:

σ_i	σ_{i+1}	b_i
+1	+1	+1
+1	−1	−1
−1	+1	−1
−1	−1	+1

We need an addit factor of 2 to distinguish degenerate states!

For N spins, we need N-1 bond variables (and a factor of 2) to visit all states

$$Q_N = \sum_{\{\sigma_i = \pm 1\}}^{N} e^{\beta J \sum_i \sigma_i \sigma_{i+1}} = 2 \sum_{\{b_i = \pm 1\}}^{N-1} e^{\beta J \sum_i b_i}$$

$$= 2 \sum_{\{b_i = \pm 1\}} e^{\beta J b_1} e^{\beta J b_2} \cdots e^{\beta J b_{N-1}}$$

$$Q_N = 2\sum_{b_1=\pm1} e^{\beta J b_1} \sum_{b_2=\pm1} e^{\beta J b_2} \cdots \sum_{b_{N-1}=\pm} e^{\beta J b_{N-1}}$$

$$= 2\left(e^{\beta J} + e^{-\beta J}\right)^{N-1} \qquad = 2(2\cosh\beta J)^{N-1}$$

$$\boxed{Q_N = 2(2\cosh\beta J)^{N-1}}$$

Next, without periodic boundaries or dual-lattice bond variables: $\mathcal{H} = -J\sum_{n=1}^{N-1}\sigma_n\sigma_{n+1}$

$$Q_N = \sum_{\{\sigma_i=\pm1\}} e^{\beta J\sum_{i=1}^{N-1}\sigma_i\sigma_{i+1}}$$

$$= \sum_{\{\sigma_i=\pm1\}} e^{\beta J\sigma_1\sigma_2} e^{\beta J\sigma_2\sigma_3} \cdots e^{\beta J\sigma_{N-2}\sigma_{N-1}} e^{\beta J\sigma_{N-1}\sigma_N}$$

Do $\sigma_N = \pm1$ first:

$$Q_N = \sum_{\{\sigma_i=\pm1\}} e^{\beta J\sigma_1\sigma_2} e^{\beta J\sigma_2\sigma_3} \cdots e^{\beta J\sigma_{N-2}\sigma_{N-1}} \underbrace{\left(e^{\beta J\sigma_{N-1}} + e^{-\beta J\sigma_{N-1}}\right)}$$

If $\sigma_{N-1} = +1$ $\quad \left(e^{\beta J} + e^{-\beta J}\right)$

If $\sigma_{N-1} = -1$ $\quad \left(e^{-\beta J} + e^{\beta J}\right)$

$= 2\cosh\beta J$

$\therefore$

$$Q_N = 2\cosh\beta J \sum_{\{\sigma_i\}} e^{\beta J\sum_{n=1}^{N-2}\sigma_n\sigma_{n+1}}$$

$$Q_N = 2\cosh\beta J \; Q_{N-1}$$

We can continue the sequence all the way down to Q_1:

$$Q_N = (2\cosh \beta J)^{N-1} \, Q_1$$

$$= (2\cosh\beta J)^{N-1} \sum_{\sigma_1=\pm 1} \sum_{\sigma_2=\pm 1} e^{\beta J \sigma_1 \sigma_2}$$

These last terms we can do explicitly:

$$Q_N = (2\cosh\beta J)^{N-1} \Big(\underbrace{e^{\beta J}}_{\substack{\sigma_1=1\\ \sigma_2=1}} + \underbrace{e^{-\beta J}}_{\substack{\sigma_1=1\\ \sigma_2=-1}} + \underbrace{e^{-\beta J}}_{\substack{\sigma_1=-1\\ \sigma_2=1}} + \underbrace{e^{+\beta J}}_{\substack{\sigma_1=-1\\ \sigma_2=-1}} \Big)$$

$$Q_N = (2\cosh\beta J)^{N-1} \; 2\left(e^{\beta J} + e^{-\beta J}\right)$$

$$\boxed{Q_N = 2\cdot(2\cosh\beta J)^N} \longleftarrow \text{without periodic boundaries}$$

$$\boxed{Q_N = 2\cdot(2\cosh\beta J)^{N-1}} \longleftarrow \text{with periodic boundaries}$$

Free energies:

$$A(N,V,T) = -k_B T \ln Q_N = -k_B T \left[\ln 2 + N \ln 2\cosh\beta\right.$$

$$A(N,V,T) = -kT\ln 2 - NkT\ln\left[2\cosh\beta J\right)$$

$$\langle E\rangle = -kT^2 \frac{\partial \ln Q}{\partial T} = \frac{\partial \ln Q}{\partial \beta} = N\frac{1}{2\cosh\beta J}\cdot 2\sinh\beta J \cdot$$

$$\langle E\rangle = NJ\tanh\beta J$$

$$C_V = \frac{\partial\langle E\rangle}{\partial T} = \frac{-J^2 N}{kT^2}\left(\operatorname{sech}\beta J\right)^2$$

With a field

$$\mathcal{H} = -H\sum_{i=1}^{N}\sigma_i - J\sum_{i=1}^{N-1}\sigma_i\sigma_{i+1} \overset{\text{PBC}}{=} -\frac{H}{2}\sum_{i=1}^{N}(\sigma_i+\sigma_{i+1}) - J\sum_{i=1}^{N}\sigma_i\sigma_{i+1}$$

$$Q = \sum_{\{\sigma_i=\pm1\}} e^{\beta J\sum_i \sigma_i\sigma_{i+1} + \frac{\beta H}{2}\sum_i(\sigma_i+\sigma_{i+1})}$$

Define a transfer matrix $\underline{P}$:

$$\langle\sigma|P|\sigma'\rangle = e^{\beta[J\sigma\sigma' + H(\sigma+\sigma')/2]}$$

$$\langle 1|P|1\rangle = e^{\beta(J+H)}$$
$$\langle 1|P|-1\rangle = e^{-\beta J}$$
$$\langle -1|P|1\rangle = e^{-\beta J}$$
$$\langle -1|P|-1\rangle = e^{\beta(J-H)}$$

$$\underline{P} = \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ +e^{-\beta J} & e^{\beta(J-H)} \end{pmatrix}$$

connects states of two adjacent spins

$$Q = \sum_{\{\sigma_i=\pm1\}} \langle\sigma_1|P|\sigma_2\rangle\langle\sigma_2|P|\sigma_3\rangle\langle\sigma_3|P|\sigma_4\rangle \cdots \langle\sigma_N|P|\sigma_1\rangle$$

Using closure relation $\sum_{\sigma_i}|\sigma_i\rangle\langle\sigma_i| = 1$

$$Q = \sum_{\sigma_1=\pm1}\langle\sigma_1|P^N|\sigma_1\rangle = \mathrm{Tr}\left[\underline{P}^N\right]$$

To carry out the trace, we first diagonalize P:

A brief interlude on 2x2 matrices

$$\underline{\underline{A}} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \qquad \underline{\underline{B}} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$[A \cdot B]_{ij} = \sum_{k=1}^{2} A_{ik} B_{kj}$$

$$\mathrm{tr}[\underline{\underline{A}}] = A_{11} + A_{22} = \sum_{k=1}^{2} A_{kk}$$

The Trace is conserved for cyclic permutations

$$\mathrm{tr}[ABC] = \mathrm{tr}[CAB] = \mathrm{tr}[BCA]$$

but not for acyclic permutations:

$$\mathrm{tr}[ABC] \neq \mathrm{tr}[BAC]$$

Diagonalization

$$\underline{\underline{M}} = \underline{\underline{U}}^T \cdot \underline{\underline{A}}\, \underline{\underline{U}}$$

for an arbitrary square matrix $\underline{\underline{A}}$, there is a Unitary transformation which results in a diagonal matrix $\underline{\underline{M}}$

$$\underline{\underline{M}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

λ_1 & λ_2 are the eigenvalues of $\underline{\underline{A}}$

$\underline{\underline{U}}$ = matrix of unit eigenvectors of $\underline{\underline{A}}$

Columns of $\underline{\underline{U}}$ are eigenvectors of $\underline{\underline{A}}$

$$\left.\begin{aligned} \underline{\underline{A}} \cdot u_1 &= \lambda_1 u_1 = \lambda_1 \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} \\ \underline{\underline{A}} \cdot u_2 &= \lambda_2 u_2 = \lambda_2 \begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix} \end{aligned}\right\} \longrightarrow \underline{\underline{U}} = \begin{pmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{pmatrix}$$

The diagonalization transform is a <u>unitary</u> similarity transform

$$\underline{\underline{U}}^T = \underline{\underline{U}}^{-1}$$

$$\underline{\underline{U}}^T \cdot \underline{\underline{U}} = \underline{\underline{U}}^{-1} \underline{\underline{U}} = \underline{\underline{I}}$$

The identity matrix

Now, consider: $\quad Tr\left[P^N\right] = \sum_k \left[P^N\right]_{kk}$

hard to determine

Suppose we diagonalize $\underline{\underline{P}}$ first,

$$\underline{\underline{M}} = \underline{\underline{U}}^T \cdot \underline{\underline{P}} \cdot \underline{\underline{U}}$$

$$\underline{\underline{M}}^N = \left(\underline{\underline{U}}^T \cdot \underline{\underline{P}} \cdot \underline{\underline{U}}\right)\left(\underline{\underline{U}}^T \cdot \underline{\underline{P}} \cdot \underline{\underline{U}}\right)\left(\underline{\underline{U}}^T \cdot \underline{\underline{P}} \cdot \underline{\underline{U}}\right) \cdots$$

$$= \underline{\underline{U}}^T \cdot \underline{\underline{P}} \cdot \left(\underline{\underline{U}}\,\underline{\underline{U}}^T\right) \cdot \underline{\underline{P}} \cdot \left(\underline{\underline{U}}\,\underline{\underline{U}}^T\right) \cdot \underline{\underline{P}} \cdot \left(\underline{\underline{U}}\,\underline{\underline{U}}^T\right) \cdots$$

$\underline{\underline{U}}^T = \underline{\underline{U}}^{-1}$

$$= \underline{\underline{U}}^T \cdot \underline{\underline{P}} \cdot \underline{\underline{I}} \cdot \underline{\underline{P}} \cdot \underline{\underline{I}} \cdot \underline{\underline{P}} \cdot \underline{\underline{I}} \cdots$$

$$\underline{\underline{M}}^N = \underline{\underline{U}}^T \cdot \underline{\underline{P}}^N \cdot \underline{\underline{U}}$$

So: $\quad tr\left[\underline{\underline{M}}^N\right] = tr\left[\underline{\underline{U}}^T \cdot \underline{\underline{P}}^N \cdot \underline{\underline{U}}\right]$

cyclic permutation

$$= tr\left[\underline{\underline{U}}\,\underline{\underline{U}}^T P^N\right]$$

$$tr\left[M^N\right] = tr\left[P^N\right]$$

$$\therefore \quad Q_N = tr\left[\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^N\right] = tr\left[\begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix}\right] = \lambda_1^N + \lambda_2^N$$

Now, back to the problem at hand:

$$\mathcal{H} = \sum_n \left[-J\sigma_n\sigma_{n+1} - \frac{H}{2}(\sigma_n + \sigma_{n+1}) \right]$$

$$Q_N = \sum_{\sigma_1=\pm1} \cdots \sum_{\sigma_N=\pm1} \langle\sigma_1| \underbrace{\left[e^{\beta J\sigma_1\sigma_2 + \frac{\beta H}{2}(\sigma_1+\sigma_2)} \right]}_{\underline{\underline{P}}} |\sigma_2\rangle\langle\sigma_2| \ldots$$

$\underline{\underline{P}}$ = transfer matrix connecting σ_1 to σ_2

$$\begin{pmatrix} e^{\beta J+\beta H} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J-\beta H} \end{pmatrix}$$

$$Q_N = \mathrm{Tr}[\underline{\underline{P}}^N] = \mathrm{Tr}[\underline{\underline{U}}^{\dagger}\underline{\underline{M}}^N\underline{\underline{U}}] = \mathrm{Tr}[\underline{\underline{M}}^N]$$

$$= M_{11}^N + M_{22}^N = \lambda_1^N + \lambda_2^N \Leftarrow$$ λ_1 & λ_2 are eigen values of P

$$\det[\underline{\underline{P}} - \lambda\underline{\underline{I}}] = 0 \Longrightarrow \begin{vmatrix} e^{\beta J+\beta H}-\lambda & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J-\beta H}-\lambda \end{vmatrix} = 0$$

$$\left(e^{\beta J+\beta H}-\lambda\right)\left(e^{\beta J-\beta H}-\lambda\right) - e^{-2\beta J} = 0$$

$$e^{2\beta J} - \lambda\left(e^{\beta J+\beta H} + e^{\beta J-\beta H}\right) + \lambda^2 - e^{-2\beta J} = 0$$

$$\left(e^{2\beta J} - e^{-2\beta J}\right) - e^{\beta J}\lambda\left(e^{\beta H} + e^{-\beta H}\right) + \lambda^2 = 0$$

25-4.4

$$2\sinh 2\beta J - e^{\beta J}\lambda(2\cosh\beta H) + \lambda^2 = 0$$

$$\lambda = \frac{e^{\beta J}2\cosh\beta H \pm \sqrt{e^{2\beta J}4\cosh^2\beta H - 8\sinh 2\beta J}}{2}$$

$$\lambda = e^{\beta J}\cosh\beta H \pm \sqrt{e^{2\beta J}\cosh^2\beta H - 2\sinh(2\beta J)}$$

$$= e^{\beta J}\cosh\beta H \pm \sqrt{e^{2\beta J}\cosh^2\beta H - e^{2\beta J} + e^{-2\beta J}}$$

$$= e^{\beta J}\left(\cosh\beta H \pm \sqrt{\cosh^2\beta H - 1 + e^{-4\beta J}}\right)$$

$$\boxed{\lambda_\pm = e^{\beta J}\left(\cosh\beta H \pm \sqrt{\sinh^2\beta H + e^{-4\beta J}}\right)}$$

$$Q_N = \lambda_+^N + \lambda_-^N$$ ← one will always be larger

$1.1^N + 0.9^N$

← will dominate as $N \to \infty$

$$Q_N \approx \left(e^{\beta J}\left(\cosh\beta H + \sqrt{\sinh^2\beta H - e^{-4\beta J}}\right)\right)^N$$

$$A \approx -Nk_BT\ln\left[e^{\beta J}\cosh\beta H + \left(e^{2\beta J}\sinh^2\beta H + e^{-2\beta J}\right)^{1/2}\right]$$

$$m = \langle\sigma_i\rangle = \frac{-1}{N}\frac{\partial A}{\partial H} = \frac{1}{\beta\lambda_+}\frac{\partial\lambda_+}{\partial H}$$

25-4.5

$$m = \frac{\sinh(\beta H)}{\sqrt{\sinh^2 \beta H + e^{-4\beta J}}}$$

$\therefore$ when $H = 0$, there is no spontaneous magnetization at any temperature in 1 dimension

In 2D, there is!

Experimental tie: magnetic susceptibility $\chi = \frac{\partial \langle m \rangle}{\partial H}$